Displayed Categories in Lean4

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Chapter 1

Displayed Categories

In this section we develop some of the definitions and lemmas related to displayed categories that we will use in the rest of the notes.

Definition 1.0.1 (Displayed Structure)**.** A **displayed structure** over a category $\mathbf{C}$ consists of

- A type family $D : \mathbf{C} \to \mathsf{Type}$ assigning to each object $c : \mathbf{C}$ a type $D(c)$ of 'objects over c';
- For each morphism $f : I \to J$ of $\mathbf{C}$, and objects $x : D(I)$ and $y : D(J)$, a type of 'morphisms from x to y over f', denoted $\mathsf{hom}_f(x, y)$ or $x \to_f y$;
- For each $c : \mathbf{C}$ and $x : D(c)$, a morphism $1_x : x \to_{1_c} x$;
- For all morphisms $f : I \to J$ and $g : J \to K$ in $\mathbf{C}$ and objects $x : D(I)$ and $y : D(J)$ and $z : D(K)$, a function

  $$\mathsf{hom}_f(x, y) \to \mathsf{hom}_g(y, z) \to \mathsf{hom}_{f \circ g}(x, z)\,,$$

  denoted like ordinary composition by $\bar{f} \mapsto \bar{g} \mapsto \bar{f} \circ \bar{g} : x \to_{f \circ g} z$, where $\bar{f} : x \to_f y$ and $\bar{g} : y \to_g z$.

Definition 1.0.2. A **displayed category** $\mathbb{D}$ over a category $\mathbf{C}$ is a displayed structure such that the following conditions hold:

- (Left unitality) $1_x \circ \bar{f} =_* \bar{f}$ for all $x : D(I)$ and $\bar{f} : x \to_f y$;
- (Right unitality) $\bar{f} \circ 1_y =_* \bar{f}$ for all $y : D(J)$ and $\bar{f} : x \to_f y$;
- (Associativity) $\bar{f} \circ (\bar{g} \circ \bar{h}) =_* (\bar{f} \circ \bar{g}) \circ \bar{h}$ for all $\bar{f} : x \to_f y$, $\bar{g} : y \to_g z$ and $\bar{h} : z \to_h w$.

In above, the relations $=_*$ are *dependent* equalities, over equalities of morphisms in $\mathbf{C}$. For instance, the right unit axiom $\bar{f} \circ 1_y =_* \bar{f}$ is over the ordinary right unit axiom $f \circ 1_b = f$ of $\mathbf{C}$.