CARGO NOTES

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⚠ Looking at the last page should convince you to read all pages.

Kan Extensions

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \quad \rightsquigarrow \quad \mathrm{Fun}(\mathcal{B},\mathcal{E}) \xrightarrow{F^*} \mathrm{Fun}(\mathcal{A},\mathcal{E})$$

$$\begin{tikzcd} G \arrow[d, Rightarrow, "\alpha"'] & & G\circ F \arrow[d, "\alpha\circ F"] \\ G' & \longmapsto & G'\circ F \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] \arrow[dr, phantom, "=" near start, shift left=3] & \mathcal{B} \arrow[d, dotted, "K"] \\ & \mathcal{E} \end{tikzcd}$$

Q. $\exists K$ s.t. $K\circ F = G$?

A: Not necessarily.

Could be

$$\begin{tikzcd} h\neq k \arrow[r, mapsto] \arrow[dr] & Fh=Fk \arrow[d, dotted, "/" marking] \\ & Gh\neq Gk \end{tikzcd}$$

Resolution:

So we look for the best approximation.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}" near end, shift left=3] & \mathcal{B} \arrow[d, dotted, "\mathrm{Lan}_F^G"] \\ & \mathcal{E} \end{tikzcd}$$

(Left Kan Extension)

Thm/Def:

(Lan_F^G, η) is the initial object in the category G/F^*:

$$\begin{tikzcd} G/F^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\eta}{\Nearrow}"] & \mathrm{Fun}(\mathcal{B},\mathcal{E}) \arrow[d, "F^*"] \\ 1 \arrow[r, "G"'] & \mathrm{Fun}(\mathcal{A},\mathcal{E}) \end{tikzcd}$$

Prop. $\mathrm{Nat}(\mathrm{Lan}_F^G, K) \cong \mathrm{Nat}(G, K \circ F)$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{B} \arrow[d, dashed, "\mathrm{Lan}_F^G" description] \arrow[d, bend left=60, "K"] \\
 & \mathcal{E}
\arrow[from=1-1, to=2-2, phantom, shift left=4, "\overset{\eta}{\Rightarrow}"]
\arrow[from=1-2, to=2-2, phantom, shift left=5, "\Rightarrow"]
\end{tikzcd}$$

$\forall\ K : \mathcal{B} \to \mathcal{E}$ and

$\gamma : G \Rightarrow K \circ F$

$\exists!\ \tilde{\gamma} : \mathrm{Lan}_F^G \Rightarrow K$

s.t. $(\tilde{\gamma} \cdot F) \circ \eta = \gamma$

$$\begin{tikzcd}
 & G \arrow[dl, "\eta"'] \arrow[dr, "\gamma"] & \\
\mathrm{Lan}_F^G \circ F \arrow[rr, dashed, "\tilde{\gamma} \cdot F"'] & = & K \circ F
\end{tikzcd}$$

Right Kan Extension.

Def. $(\mathrm{Ran}_F^G, \epsilon)$ is the terminal object in the comma category F^*/G.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{B} \arrow[d, "?"] \\
 & \mathcal{E}
\end{tikzcd}$$

$$\begin{tikzcd}
F^*/G \arrow[r] \arrow[d] & 1 \arrow[d, "G"] \\
\mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[r, "F^*"'] \arrow[ur, phantom, "\overset{\epsilon}{\Rightarrow}"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{B} \arrow[d, "\mathrm{Ran}_F^G"] \\
 & \mathcal{E}
\arrow[from=1-2, to=2-1, phantom, "\overset{\epsilon}{\Leftarrow}"]
\end{tikzcd}$$

By terminality of $(\mathrm{Ran}_F^G, \epsilon)$ we have:

If
$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{B} \arrow[d, "H"] \arrow[dl, phantom, "\overset{\delta}{\Leftarrow}"] \\
& \mathcal{E}
\end{tikzcd}$$
then $\exists!\ \tilde{\delta} : H \Rightarrow \mathrm{Ran}_F^G$

s.t.
$$\begin{tikzcd}
HF \arrow[rr, "\tilde{\delta}.F"] \arrow[dr, "\delta"'] & {} \arrow[d, phantom, "="] & \mathrm{Ran}_F^G \circ F \arrow[dl, "\epsilon"] \\
& G &
\end{tikzcd}$$

<u>In terms of pasting diagrams:</u>

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{B} \arrow[d, "\mathrm{Ran}_F^G"'] \arrow[d, bend left=50, "H"] \arrow[dl, phantom, "\overset{\epsilon}{\Leftarrow}"] \arrow[d, phantom, shift left=3, "\overset{\tilde{\delta}}{\Leftarrow}"] \\
& \mathcal{E}
\end{tikzcd}$$

$$\boxed{\epsilon \circ (\tilde{\delta}.F) = \delta}$$

we have the following bijections

$$\mathrm{Nat}(\mathrm{Lan}^{G}_{F}, K) \cong \mathrm{Nat}(G, K\circ F)$$

$$\mathrm{Nat}(H, \mathrm{Ran}^{G}_{F}) \cong \mathrm{Nat}(H\circ F, G)$$

Natural Transformations as Ends

Remark

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow\theta"] \end{tikzcd}$$

$$\mathrm{Nat}(F,G) \cong \int_{c\in\mathcal{C}} \mathcal{D}(Fc,Gc)$$

$$\mathcal{D}(F-,G-) : \mathcal{C}^{op}\times\mathcal{C} \longrightarrow \mathrm{Set}$$
$$(c,d) \longmapsto \mathcal{D}(Fc,Gd)$$

$$\begin{tikzcd} W \arrow[r, "\gamma_c"] \arrow[d, "\gamma_d"'] & \mathcal{D}(Fc,Gc) \arrow[d, "\mathcal{D}(1,Gf)"] \\ \mathcal{D}(Fd,Gd) \arrow[r, "\mathcal{D}(Ff,1)"'] & \mathcal{D}(Fc,Gd) \end{tikzcd}$$

$$\forall\ c \xrightarrow{f} d$$

$$\begin{tikzcd} & c \xrightarrow{f} d \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ Fc \xrightarrow{Ff} Fd & & Gc \xrightarrow{Gf} Gd \end{tikzcd}$$

Then we get $\gamma_c(w) : Fc \longrightarrow Gc$

$\forall\, c \in \mathcal{C}$, $\forall\, w \in W$ s.t. $\forall\, c \xrightarrow{f} d$ in $\mathcal{C}$

$$\begin{tikzcd}
Fc \arrow[r, "\gamma_c(w)"] \arrow[d, "Ff"'] \arrow[dr, phantom, "="] & Gc \arrow[d, "Gf"] \\
Fd \arrow[r, "\gamma_d(w)"'] & Gd
\end{tikzcd}$$

So $\gamma_-(w) \in \mathrm{Nat}(F,G)$

$\forall\, w \in W$.

$$\begin{tikzcd}
W \arrow[r, "\gamma_-"] \arrow[dr] & \mathrm{Nat}(F,G) \arrow[d, dashed, "\cong"] \\
 & \int_{c \in \mathcal{C}} \mathcal{D}(Fc, Gc)
\end{tikzcd}$$

Given by universality of end .

Prop. For $$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{B} \arrow[d, dashed] \\ & \mathcal{E} \end{tikzcd}$$

$\forall G$ (Lan_F^G, η) exists iff $\mathrm{Lan}_F \dashv F^*$.

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[r, bend right, "F^*"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[l, bend right, "\mathrm{Lan}_F"'] \end{tikzcd}$$

Similarly, $(\mathrm{Ran}_F^G, \epsilon)$ exists $\forall G$ iff

$F^* \dashv \mathrm{Ran}_F$

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{B}, \mathcal{E}) \arrow[r, bend left, "F^*"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{A}, \mathcal{E}) \arrow[l, bend left, "\mathrm{Ran}_F"] \end{tikzcd}$$

They immediately follow from the remark of next page.

Remark: $H: \mathcal{C} \to \mathcal{D}$ has a left adjoint iff the comma category d/H has an initial object $\forall d \in \mathcal{D}$.

$$\begin{tikzcd}
d/H \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, shift left=2, "H"] \arrow[d, phantom, "\dashv"] \\
1 \arrow[r, "d"'] & \mathcal{D} \arrow[u, dashed, shift left=2, "?"]
\end{tikzcd}$$

Let $\langle c_{in}, d \xrightarrow{\eta_d} Hc_{in} \rangle$ be the initial of d/H.

Define

$G(d) := c_{in}$

Prove G is a functor which is left adjoint to H: $G \dashv H$.

Also

$$\begin{tikzcd}
d \arrow[r, "\overset{\text{unit}}{\textcircled{η_d}}"] \arrow[dr, "f"'] & HG(d) \arrow[d, "H(\breve{f})"] \\
& Hc
\end{tikzcd}$$

$=$

Similarly: $H: \mathcal{C} \to \mathcal{D}$ has a right adjoint iff H/d has a terminal object.

A Remark on Dualization

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[dr, "G"'] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}" near end, shift left=3] & \mathcal{B} \arrow[d, "\mathrm{Lan}_F^G"] \\
& \mathcal{E}
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\mathcal{A}^{op} \arrow[r, "F^{op}"] \arrow[dr, "G^{op}"'] \arrow[dr, phantom, "\overset{\eta^{op}}{\Leftarrow}" near end, shift left=3] & \mathcal{B}^{op} \arrow[d, "(\mathrm{Lan}_F G)^{op}"] \\
& \mathcal{E}^{op}
\end{tikzcd}$$

But $(\mathrm{Lan}_F G)^{op} = \mathrm{Ran}_{F^{op}} G^{op}$

Note: ! In general

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "H"{name=H}] \arrow[r, bend right, "K"'{name=K}] & \mathcal{D} \arrow[Rightarrow, from=H, to=K, "\theta"]
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, bend left=50, "H^{op}"{name=H}] \arrow[r, bend right=50, "K^{op}"'{name=K}] & \mathcal{D}^{op} \arrow[Rightarrow, from=K, to=H, "\theta^{op}"]
\end{tikzcd}$$

$$Hc \xrightarrow{\theta_c} Kc$$

$$K(c) = K^{op}(c) \xrightarrow{\theta_c^{op}} H^{op}(c) = H(c)$$

Exercise: Use comma definition of Lan, Ran to prove these Claims:

Reminder

$\langle \mathrm{Lan}_F^G, \eta \rangle$ initial in G/F^*.

$\langle \mathrm{Ran}_F^G, \epsilon \rangle$ terminal in F^*/G.

Example:

(Free cocompletion) / via Kan ext.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor.

Then $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ induces

$$(F^{op})^* : Psh(\mathcal{D}) \longrightarrow Psh(\mathcal{C})$$

defined as

$$(F^{op})^* (Q) = Q \circ F^{op} \quad . \text{ so,}$$

$$(F^{op})^* (Q) \, (c) = Q(Fc) \underset{\substack{\uparrow \\ \text{Yoneda} \\ \text{Lemma}}}{\cong} Psh(\mathcal{D})(y_{Fc}, Q)$$

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "F^{op}"] \arrow[dr, "P"'] & \mathcal{D}^{op} \arrow[d, dashed, "\mathrm{Lan}_{F^{op}} P"] \\
& \mathcal{S}et
\end{tikzcd}$$

η

From the proposition proved before,

In order to prove

$\mathrm{Lan}_{F^{op}} P$ exists for every $P \in Psh(\mathcal{C})$,

we need to prove $(F^{op})^*$ has a left adjoint.

$$\begin{tikzcd}
Psh(\mathcal{C}) \arrow[rr, phantom, ""] & & Psh(\mathcal{D}) \arrow[ll, "(F^{op})^*"'] \\
& \neq & \\
\mathcal{C} \arrow[uu, "y_{\mathcal{C}}"] \arrow[rr, "F"'] & & \mathcal{D} \arrow[uu, "y_{\mathcal{D}}"']
\end{tikzcd}$$

By definition of Left Kan extension we have

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "F^{op}"] \arrow[dr, "P"'] & \mathcal{D}^{op} \arrow[d, "\mathrm{Lan}^{P}_{F^{op}}"] \arrow[dl, phantom, "\Rightarrow"] \\
 & \mathrm{Set}
\end{tikzcd}$$

If $\mathrm{Lan}^{P}_{F^{op}}$ exits for every presheaf P on $\mathcal{C}$, then $\mathrm{Lan}_{F^{op}} \dashv (F^{op})^*$

For simplicity, we denote

$$F_! = \mathrm{Lan}_{F^{op}} .$$

We will later prove that the following diagram commutes up to <u>ISO</u>:

$$\begin{tikzcd}
\mathrm{Psh}(\mathcal{C}) \arrow[r, "\mathrm{Lan}_{F^{op}}"] & \mathrm{Psh}(\mathcal{D}) \\
\mathcal{C} \arrow[u, hook, "y_{\mathcal{C}}"] \arrow[r, "F"'] & \mathcal{D} \arrow[u, hook, "y_{\mathcal{D}}"']
\end{tikzcd}$$

⚠ Some care is needed when dealing with $(F^{op})^*$.

For instance $(F^{op})^* \not\simeq (F^*)^{op}$

$$(F^{op})^* : Psh(\mathcal{D}) \longrightarrow Psh(\mathcal{C})$$

$$(F^*)^{op} : [\mathcal{D}, Sets]^{op} \longrightarrow [\mathcal{C}, Sets]^{op}$$

and $Psh(\mathcal{D}) \not\simeq [\mathcal{D}, Sets]^{op}$. For instance take $\mathcal{D} = 1$.

$$Psh(1) \cong Sets \quad \text{, and} \quad [1, Sets]^{op} \cong Sets^{op}$$

and $Sets \not\simeq Sets^{op}$

Also:

Eventhough $F : \mathcal{C} \to \mathcal{D}$, $F^{op} : \mathcal{C}^{op} \longrightarrow \mathcal{D}^{op}$ have the same effect on objects and morphism they are different from $\boxed{\text{2-categorical viewpoint}}$.

we give an explicit definition of $F_! \dashv (F^{op})^*$

$$F_! : Psh(\mathcal{C}) \longrightarrow Psh(\mathcal{D})$$

$$P \longmapsto colim\left(\int_{\mathcal{C}} P \xrightarrow{\pi} \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{y_D} Psh(\mathcal{D})\right)$$

$$F_!(P)(D) = \operatorname*{colim}_{\langle c_i, x_i\rangle \in \int_{\mathcal{C}} P} y_D(Fc_i)(D) =$$

$$\operatorname*{colim}_{\langle c_i, x_i\rangle \in \int_{\mathcal{C}} P} Hom(D, Fc_i) \cong \operatorname*{colim}_{yc_i \Rightarrow P \,\wedge\, yD \Rightarrow y(Fc_i)} \{*\}$$

The result is a quotient set

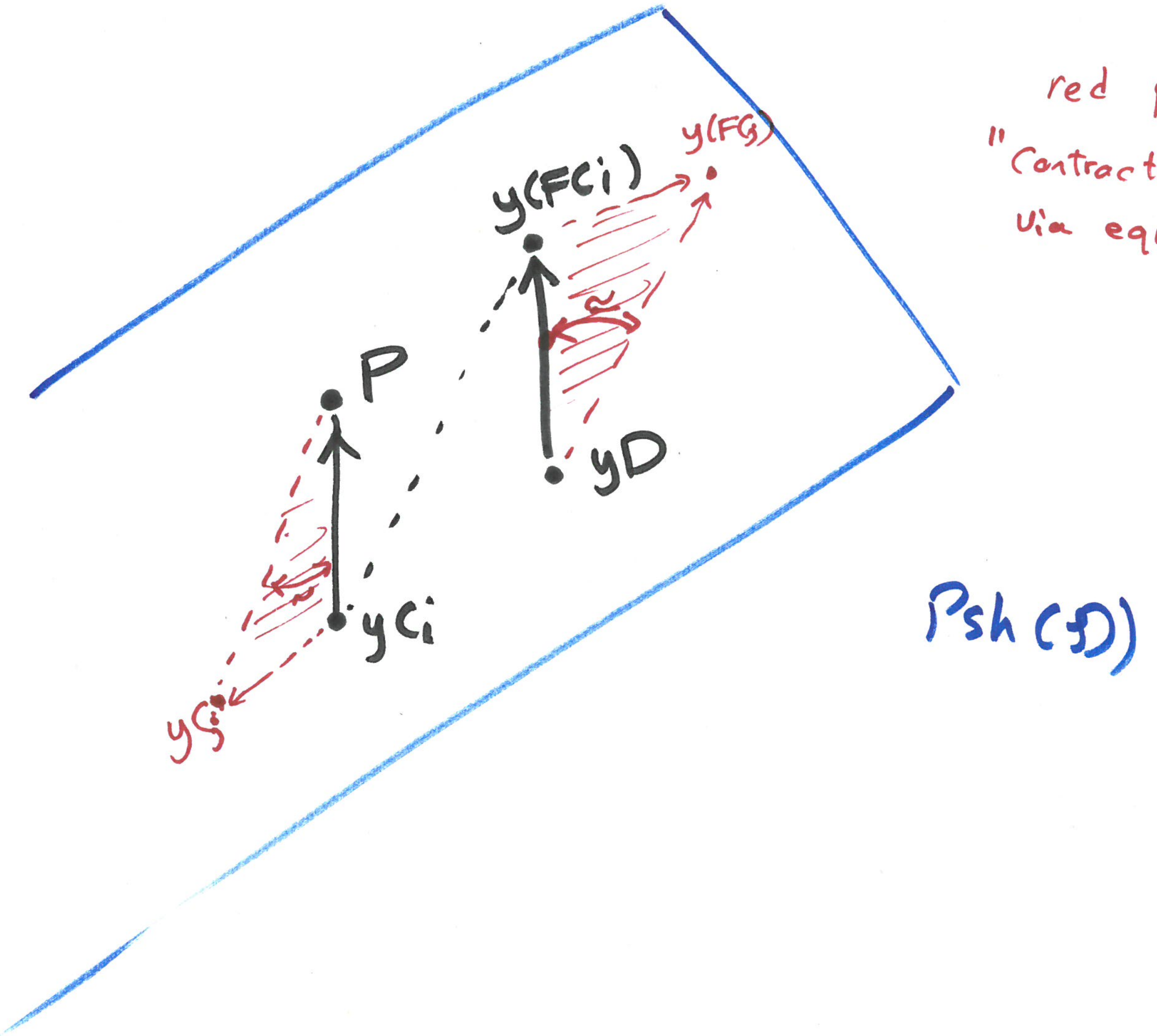
$$\coprod_{\substack{r: D \to Fc_i \\ x_i \in Pc_i}} \{*\} \Big/ \sim$$

where $\langle x_i, r: D \to Fc_i\rangle$ and $\langle x_j, r': D \to Fc_j\rangle$ are identified exactly when $\exists\, f: c_i \to c_j$ (in $\mathcal{C}$) with $x_j \circ f = x_i$, and $Ff \circ r = r'$.

Schematically, $F_!(P)(D)$ is a set whose points are the collection of pairs of arrows

$$\left\{ \langle yC_i \to P,\ yD \to y(FC_i) \rangle \right\}/_{\sim}$$

in $\mathrm{Psh}(\mathcal{D})$. (identified by $\sim$).



red parts = "Contractible via equality"

Now, let's prove that $F_!$ is indeed a left adjoint to $(F^{op})^*$.

Claim: $F_! \dashv F^*$

$$\begin{tikzcd}
P : \mathrm{Psh}(\mathcal{C}) \arrow[r, bend left, "F_!"] \arrow[r, phantom, "\perp"] & \mathrm{Psh}(\mathcal{D}) : Q \arrow[l, bend left, "F^*"]
\end{tikzcd}$$

$$P \xRightarrow{\alpha} F^*Q$$

$$\overline{\overline{\hspace{6cm}}} \quad \text{(cocompleteness of } \mathrm{Psh}(\mathcal{C})\text{)}$$

$$\operatorname{Colim}_{j \in \int_{\mathcal{C}} P} yC_j \xRightarrow{\alpha} F^*Q$$

$$\overline{\overline{\hspace{6cm}}} \quad \text{(Def. of colimit)}$$

$$\forall j \in \textstyle\int_{\mathcal{C}} P . \quad yC_j \xRightarrow{\alpha_j} F^*Q$$

$$\overline{\overline{\hspace{6cm}}} \quad \text{(Yoneda)}$$

$$\forall j \in \textstyle\int_{\mathcal{C}} P . \quad \alpha_j \in F^*Q(C_j) = Q(FC_j)$$

$$\overline{\overline{\hspace{6cm}}} \quad \text{(Yoneda)}$$

$$\forall j \in \textstyle\int_{\mathcal{C}} P \quad y_{FC_j} \xRightarrow{\alpha_j} Q$$

$$\overline{\overline{\hspace{6cm}}}$$

$$F_!P := \operatorname{Colim}_{j \in \int_{\mathcal{C}} P} y(FC_j) \xRightarrow{\operatorname{colim} \alpha_j} Q$$

Defining $F_! \dashv (F^{op})^*$ in an alternative way

First observe that

$$\boxed{\underset{D/F}{\mathrm{Colim}}\; y_{C_0} \circ \pi_0 \cong y_{FC_0}(D)}$$

where $F : \mathcal{C} \to \mathcal{D}$

$D \in ob(\mathcal{D})$, $C_0 \in ob(\mathcal{C})$

and $\pi_0 : \mathcal{D}/F \longrightarrow \mathcal{C}$ in the following diagram of comma object:

$$\begin{tikzcd}
D/F \arrow[r, "\pi_0"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \mathcal{C} \arrow[d, "F"] \\
1 \arrow[r, "D"'] & \mathcal{D}
\end{tikzcd}$$

Proof:

$$\operatorname{Colim}_{D/F} \; y_{C_0} \circ \bar{\pi}_0 =$$

$$\operatorname{colim}_{\langle C,\, D \xrightarrow{f} FC \rangle} \; y_{C_0}(C) = \operatorname{Colim}_{\langle C, f\rangle} \; \operatorname{Hom}(C_0, C)$$

$$\cong \operatorname{Hom}(D, FC_0) \cong y_{FC_0}(D)$$

$$\begin{tikzcd}
 & \operatorname{Hom}(\bar{\pi}_0\langle C, D \xrightarrow{f} FC\rangle, C_0) \arrow[d, equal] & & \operatorname{Hom}(\bar{\pi}_0\langle C', D \xrightarrow{g} FC'\rangle, C_0) \arrow[ll, "{\operatorname{Hom}(\pi_0 \ell,\, id)}"'] \arrow[d, equal] & \\
\cdots & \operatorname{Hom}(C, C_0) \arrow[l] \arrow[dr, dashed] & & \operatorname{Hom}(C', C_0) \arrow[ll, "- \circ \ell"'] \arrow[dl, dashed] & \cdots \arrow[l] \\
 & C \xrightarrow{t} C_0 \arrow[ddr, bend right, <->] & \operatorname{Hom}(D, FC_0) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \\
 & & & & \\
 & & D \xrightarrow{f} FC \xrightarrow{F(t)} FC_0 & &
\end{tikzcd}$$

Also, $\operatorname{Hom}(D, FC_0)$ is the initial cocone.

This and next page = motivation for definition of $F_!$

Since we desire $F_! \dashv (F^{op})^*$ we can write local isomorphism of adjunction for representables:

$$\mathrm{Hom}(F_!(y_{C_0}), y_D) \cong \mathrm{Hom}(y_{C_0}, F^* y_D)$$

$$\cong \mathrm{Hom}(y_{C_0}, y_D \circ F) \cong$$

$$(y_D \circ F)(C_0) = y_D(FC_0) =$$

$$\mathrm{Hom}(FC_0, D) \cong \mathrm{Hom}(y_{FC_0}, y_D)$$

By the Yoneda lemma, we need $\boxed{F_!(y_{C_0}) \cong y_{FC_0}}$

Now, for every $P \in \mathrm{Psh}(\mathcal{C})$, we try to define $F_!(P)$:

Since we want $F_! \dashv (F^{op})^*$, $F_!$ would better be cocontinuous. So:

$$F_!(P)(D) = F_!\Big(\operatorname*{Colim}_{\langle C, x\in PC\rangle \in \int_{\mathcal{C}} P} yC\Big)(D) \cong$$

$$\operatorname*{Colim}_{\langle C, x\in PC\rangle} F_!(yC)(D) \cong$$

$$\operatorname*{Colim}_{\langle C, x\in PC\rangle} y_{FC}(D) =$$

$$\operatorname*{Colim}_{\langle C, x\in PC\rangle} \mathrm{Hom}(D, FC) \cong$$

$$\operatorname*{Colim}_{\langle C, x\in PC\rangle} \operatorname*{Colim}_{D\to FC} \{*\} \cong$$

$$\operatorname*{Colim}_{\langle C, D\to FC\rangle} \operatorname*{Colim}_{x\in PC} \{*\} \cong \operatorname*{Colim}_{\langle C, D\to FC\rangle \in D/_F} P(C)$$

$$= \operatorname*{Colim}_{D/F} P\circ\pi$$

The two definitions agree:

① $F_!(P)(D) := \mathrm{Colim}\left(\int_{\mathcal{C}} P \xrightarrow{\pi} \mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{y} \mathrm{Psh}(\mathcal{D})\right)(D)$

② $F_!(P)(D) = \underset{D/F}{\mathrm{colim}} \; P \circ \pi$

we just need to check equality on representables:

① $F_!(yC_0) = \underset{C \to C_0 \in \mathcal{C}/C_0}{\mathrm{colim}} \; yF(C) \cong y(FC_0)$

Noting that $\int_{\mathcal{C}} yC_0 = \mathcal{C}/C_0$

category of elements $\leftsquigarrow$ $\int_{\mathcal{C}} yC_0$; $\mathcal{C}/C_0 \rightsquigarrow$ slice category

② $F_!(yC_0)(D) = \underset{D/F}{\mathrm{colim}} \; y_{C_0} \circ \pi_0 \cong y_{FC_0}(D)$

$\cong$ proved before

So $F_!(yC_0) \cong y_{FC_0}$

Note that functoriality of $F_!$ follows from functoriality of Colim(-) functor.

We checked before

$$y_{\mathcal{D}} \circ F \cong F_! \circ y_{\mathcal{C}}$$

$$\begin{tikzcd}
Psh(\mathcal{C}) \arrow[r, shift left, "F_!"] \arrow[r, phantom, "\perp"] & Psh(\mathcal{D}) \arrow[l, shift left, "(F^{op})^*"] \\
\mathcal{C} \arrow[u, hook, "y_{\mathcal{C}}"] \arrow[r, "F"'] \arrow[ur, phantom, "\cong"] & \mathcal{D} \arrow[u, hook', "y_{\mathcal{D}}"']
\end{tikzcd}$$

Note that

$y_{\mathcal{C}} \neq (F^{op})^* \circ y_{\mathcal{D}} \circ F$ since

$yc(c') = \mathrm{Hom}(c', c)$ but,

$(F^{op})^* \circ y_{\mathcal{D}} \circ F(c)(c') = (F^{op})^*(y(Fc))(c')$

$= \mathrm{Hom}(Fc', Fc)$

Universality of Kan Ext.

we showed $F_! := \mathrm{Lan}_{F^{op}} \dashv (F^{op})^*$

and $F_!(yC) \cong y(FC)$

Indeed for representable presheaves we can write the diagram of left Kan ext. explicitly:

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "F^{op}"] \arrow[dr, "yC"'] \arrow[dr, phantom, shift left=4, "\overset{\eta}{\Rightarrow}"] & \mathcal{D}^{op} \arrow[d, "\mathrm{Lan}_{F^{op}}^{yC} = F_!(yC)"] \\
 & \mathrm{Set}
\end{tikzcd}$$

where

$$\eta_X : yC(X) \Rightarrow F_!(yC) \circ F^{op}(X)$$

$$\begin{array}{rl} \eta_X : \mathrm{Hom}_{\mathcal{C}}(X,C) \longrightarrow & Y(FC)\,(FX) \\ & \quad \| \\ & \mathrm{Hom}_{\mathcal{D}}(FX,FC) \end{array}$$

so η_X can defined as

$$F_{X,C} : \mathrm{Hom}_{\mathcal{C}}(X,C) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FX,FC)$$

Universality of left Kan extension

Suppose $\gamma : yC \Rightarrow Q \circ F^{op}$

in
$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "F^{op}"] \arrow[dr, "yC"', ""{name=A}] & \mathcal{D}^{op} \arrow[d, "Q"] \\ & \mathrm{Set} \arrow[phantom, from=A, to=1-2, "\overset{\gamma}{\Rightarrow}"] \end{tikzcd}$$

then $\gamma_X : \mathrm{Hom}(X, C) \longrightarrow Q(FX)$. (natural in X)

we want to find $\tilde{\gamma} : F_!(yC) \Rightarrow Q$

s.t. $(\tilde{\gamma} . F^{op}) \circ \eta = \gamma$.

Note that $\tilde{\gamma} : F_!(yC) \Rightarrow Q$ amounts

to $\tilde{\gamma}_D : \mathrm{Hom}(D, FC) \longrightarrow QD$

natural in D.

Define $\tilde{\gamma}(f) := Q(f)(\gamma_C(id_C))$

$\forall f : D \to FC$.

Universality for any presheaf $P : \mathcal{C} \to \mathrm{Set}$ can now be easily deduced.

Examples in posets (trivial examples)

toposes must be compared with locales. They are both "geometric", but posets/frames are "algebraic" in nature.

$$\begin{tikzcd} \text{Toposes} \arrow[r, leftrightsquigarrow] & \text{locales} \arrow[r, bend left, "\text{op}"] & \text{Frames} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "!"] & \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X} \arrow[r, "!"] & 1 \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[r, "!"] & \mathcal{O}(\mathcal{X}) \end{tikzcd}$$

terminal topos

terminal locale

initial frame

$$\begin{tikzcd} \mathbf{Set} \arrow[r, shift right, "f_*"'] & \mathcal{E} \arrow[l, shift right, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "x"] & \mathcal{X} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}(\mathcal{X}) \arrow[r, "x"] & 2 \end{tikzcd}$$

a point of $\mathcal{E}$

a point of locale $\mathcal{X}$.

a frame morphism.

So, we are going to replace

topos

Sets with frm 2 in our previous discussion.

Important Lemm/Factoid

Note!

The poset P is a frame iff y_P has a lex left adjoint.

P, Q: posets

$$\begin{tikzcd}
2^{P^{op}} \arrow[r, dashed, "f_!"] \arrow[d, dashed, bend right] & 2^{Q^{op}} \arrow[d, dashed, bend right, "\dashv"] \\
P \arrow[u, "y_P"'] \arrow[r, "f"'] & Q \arrow[u, "y_Q"']
\end{tikzcd}$$

$\boxed{\text{Note}}$

(i) $d : P^{op} \to 2 \longleftrightarrow D = \{p \in P \mid d(p) = 1\}$

✓ a ~~frame~~ poset morphism (so d: monotone)

$\rightarrow$ downset

(ii) $y_P(p) \longleftrightarrow \downarrow(p)$: principal downset

(iii) $f_!(\downarrow(p)) = \downarrow(fp)$

by definition of $f_!$

(iv) $2^{P^{op}}$ is complete and cocomplete.

In particular:

$d \wedge d'(p) = d(p) \wedge d'(p) = 1 \Leftrightarrow p \in D \cap D'$

Suppose P and Q are meet semilattices and f preserves meets.

Then

$$f_!(D \cap D') = f_!\Big(\bigcup_{p \in D} \downarrow(p) \cap \bigcup_{p' \in D'} \downarrow(p') \Big)$$

$$= f_!\Big(\bigcup_{p,p'} \downarrow(p \wedge p') \Big) = \bigcup_{p,p'} f_!(\downarrow(p \wedge p')) =$$

$$\bigcup_{p,p'} \downarrow(f(p \wedge p')) = \bigcup_{p \in D} \downarrow(fp) \cap \bigcup_{p' \in D'} \downarrow(fp')$$

$$= f_!\Big(\bigcup_{p \in D} \downarrow(p) \Big) \cap f_!\Big(\bigcup_{p' \in D'} \downarrow(p') \Big)$$

$$= f_!(D) \cap f_!(D')$$

So $f_!$ also preserves meets.

($f_!$ is lex)

The Majesty of 2-categories

The operation
$$\begin{tikzcd}
\mathcal{C} \arrow[r, mapsto] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] & \mathrm{Psh}(\mathcal{C}) \arrow[d, bend right, "f_!"'] \arrow[d, bend left, "g_!"] \arrow[d, phantom, "\overset{\gamma_!}{\Rightarrow}"] \\
\mathcal{D} \arrow[r, mapsto] & \mathrm{Psh}(\mathcal{D})
\end{tikzcd}$$

defines a 2-functor

$$\begin{tikzcd}
\mathrm{Cat} \arrow[r, "\mathrm{Psh}"] \arrow[dr, dashed, "?"'] & \mathrm{CAT} \\
& \mathrm{Topos} \arrow[u]
\end{tikzcd}$$

The 2-category of small categories

The meta-2-category of categories

which factors through the 2-cat of toposes if we restrict Cat to the 2-category of sites and lex functors.

Claim. The 2-functor Psh sends monads to monads.

Proof: A monad $T : \mathcal{C} \to \mathcal{C}$ (on a small category $\mathcal{C}$)

is just a lax 2-functor

$$pt \xrightarrow{T} Cat.$$

Composing this with $Cat \xrightarrow{Psh} CAT$ gives another lax functor,

$$pt \xrightarrow{T} Cat \xrightarrow{Psh} CAT$$

which is a monad in CAT on

$Psh(\mathcal{C})$.

monads in a 2-cat $\mathcal{K}$ $\leftrightsquigarrow$ lax functors $pt \to \mathcal{K}$

A happy ending

The Yoneda embedding is

in fact a 2-embedding:
(lax)

$$\begin{tikzcd}
\mathcal{C}at \arrow[d, bend right=50, "\mathrm{inc}"'] \arrow[d, bend left=50, "Psh"] \arrow[d, phantom, "\overset{y}{\Rightarrow}"] \\
\mathcal{CAT}
\end{tikzcd}$$

so $y_{\mathcal{C}} : \underset{\mathcal{C}}{\underset{\parallel}{\mathrm{inc}(\mathcal{C})}} \longrightarrow Psh(\mathcal{C})$

for every small (locally) category $\mathcal{C}$.

$\boxed{\text{Note!}}$ The isomorphism

$$\begin{tikzcd}
Psh(\mathcal{C}) \arrow[r, "F_!"] \arrow[dr, phantom, "\cong"] & Psh(\mathcal{D}) \\
\mathcal{C} \arrow[u, hook, "y_{\mathcal{C}}"] \arrow[r, "F"'] & \mathcal{D} \arrow[u, hook', "y_{\mathcal{D}}"']
\end{tikzcd}$$

emphasises/indicates laxity of y as a 2-natural transformation.