First Introduction To Simplicial Sets

The bare minimum

Sina Hazratpour

23 February 2016

The owl of Minerva takes its flight only when the shades of night are gathering.

G.W.F. Hegel, Philosophy of Right (1820), Preface

Figure: Owl of Minerva

Overview

1. The simplicial category
2. A detour to topology
3. Enters Kan: Kan complexes

The (pre)simplicial category

Definition

The ***(pre)simplicial category*** $\bar{\Delta}$ is a category whose objects are finite ordinals $\mathbf{n} = \{0, 1, \ldots, n-1\}$ and its morphisms are (weakly) monotone maps, i.e. order-preserving maps.

Remark

- We note that we get empty ordinal. The ordinal number $\mathbf{0}$ is initial and the ordinal $\mathbf{1}$ is terminal in $\bar{\Delta}$.
- It is sometimes useful to think of $\bar{\Delta}$ as a 2-category. The objects are categories $\mathbf{n} = 0 \to 1 \to \ldots \to n-1$, morphims are functors $f : \mathbf{n} \to \mathbf{m}$, and 2-cells are natural transformations. We have $f \Rightarrow g$ iff $f(i) \leq g(i)$ for every $0 \leq i \leq n-1$

$\bar{\Delta}$ as strict monoidal category

Notice that addition of ordinals is a bifunctor.

$$+ : \bar{\Delta} \times \bar{\Delta} \to \ \bar{\Delta}$$

$$\mathbf{n} + \mathbf{m} := \{0, 1, \ldots, n + m - 1\}$$

If $f : \mathbf{n} \to \mathbf{n}'$ and $g : \mathbf{m} \to \mathbf{m}'$, then $f + g : \mathbf{n} + \mathbf{m} \to \mathbf{n}' + \mathbf{m}'$.
where

$$(f + g)(i) = \begin{cases} f(i) & \text{, if } i = 0, 1, \ldots, n-1 \\ n' + g(i - n) & \text{, otherwise} \end{cases}$$

$(\Delta, +, \mathbf{0})$ is a strict monoidal category.

Some calculation in $\bar{\Delta}$

We define μ^k to be the unique arrow $\mathbf{k} \to \mathbf{1}$. So, $\mu^0 = \eta$, $\mu^1 = 1 = id_{\mathbf{1}}$ and $\mu^2 = \mu$. From the uniqueness we get equations: for instance,

$$\mu(\mu + 1) = \mu(1 + \mu) = \mu^3 \colon \mathbf{3} \to \mathbf{1}$$

Similar reasoning shows that

$$\mu^n \left(\mu^{k_{\mathbf{1}}} + \ldots + \mu^{k_n} \right) = \mu^{(k_{\mathbf{1}} + \ldots + k_n)}$$

A Simple Observation

Suppose $f : \mathbf{m} \to \mathbf{n}$ is an arrow in the category $\bar{\Delta}$.
Let m_i be the number of elements in the fibre $f^{-1}(i)$. Then, we have

$$f = \mu^{m_0} + \mu^{m_1} + \ldots + \mu^{m_n}$$

Example

Suppose $f : \mathbf{5} \to \mathbf{5}$ is given by the assignment below:

$$\begin{tikzcd}
0 \arrow[d, mapsto] & 1 \arrow[d, mapsto] & 2 \arrow[dr, mapsto] & 3 \arrow[d, mapsto] & 4 \arrow[dl, mapsto] \\
0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

Then,

$$f = \mu^1 + \mu^1 + \mu^0 + \mu^3 + \mu^0 = 1 + 1 + \eta + \mu^3 + \eta$$

Composition in $\bar{\Delta}$: by way of example

The above decomposition helps us to compose morphisms in $\bar{\Delta}$.

Example

$$
\begin{tikzcd}
\mathbf{6} \arrow[r, "f"] & \mathbf{4} \arrow[r, "g"] & \mathbf{3} \\
0 \arrow[r] & 0 \arrow[dr] & 0 \\
1 \arrow[dr] & 1 \arrow[r] & 1 \\
2 \arrow[dr] & 2 \arrow[ur] & 2 \\
3 \arrow[r] & 3 \arrow[ur] & \\
4 \arrow[ur] & &
\end{tikzcd}
$$

- $f = \mu^1 + \mu^0 + \mu^1 + \mu^3$
- $g = \mu^0 + \mu^3 + \mu^1$
- $g \circ f = \mu^0 + \mu^2 + \mu^3$

Universal monoid of $\bar{\Delta}$

- $\langle \mathbf{1}, \eta, \mu \rangle$ forms a monoid in the monoidal category $\bar{\Delta}$.

Universal monoid of $\bar{\Delta}$

- $\langle \mathbf{1}, \eta, \mu \rangle$ forms a monoid in the monoidal category $\bar{\Delta}$.
- It is *universal* in the following sense: given any monoid $\langle C, \eta', \mu' \rangle$ in a strict monoidal category $(\mathcal{C}, \otimes, I)$, there is a unique (strict) monoidal functor $F : (\bar{\Delta}, +, \mathbf{0}) \to (\mathcal{C}, \otimes, I)$ such that $F\mathbf{1} = C$, $F\eta = \eta'$, and $F\mu = \mu'$.

$$\begin{tikzcd}
\mathbf{0} \arrow[r, "\eta"] \arrow[d, mapsto, dashed] & \mathbf{1} \arrow[d, mapsto, dashed] & \mathbf{1} + \mathbf{1} \arrow[l, "\mu"'] \arrow[d, mapsto, dashed] \\
I \arrow[r, "\eta'"'] & C & C \otimes C \arrow[l, "\mu'"]
\end{tikzcd}$$

Universal monoid of $\bar{\Delta}$

- Proof of the claim:

Suppose a strict monoidal category $(\mathcal{C}, \otimes, I)$ and a monoid $\langle C, \eta', \mu' \rangle$ in it is given. Define $F(\mathbf{k}) = C^k := C \otimes C \otimes \ldots \otimes C$. In particular $F(\mathbf{0}) = I$, and $F(\mathbf{1}) = C$. Also, define

$$F(\mu^k) : C^k \to C$$

to be the k-fold multiplication in $\mathcal{C}$. For a general morphism $f : \mathbf{m} \to \mathbf{n}$ in $\bar{\Delta}$, decompose f as $f = \mu^{m_0} + \mu^{m_1} + \ldots + \mu^{m_n}$. Now, define

$$F(f) = \otimes_k F(\mu^k)$$

It is easy to check that assignment F indeed defines a strict monoidal functor $(\bar{\Delta}, \eta, \mu) \to (\mathcal{C}, \otimes, I)$. Moreover, any strict monoidal functor F with $F(\mathbf{1}) = C$, $F(\eta) = \eta'$, and $F(\mu) = \mu'$ is uniquely determined essentially due to equations such as $\mu(\mu + 1) = \mu(1 + \mu) = \mu^3 : \mathbf{3} \to \mathbf{1}$ and $f = \mu^{m_0} + \mu^{m_1} + \ldots + \mu^{m_n}$.

Injectives and surjectives

Question

How many injective monotone $f : \mathbf{n} \to \mathbf{n+1}$ are there?

Injectives and surjectives

Question

How many injective monotone $f : \mathbf{n} \to \mathbf{n+1}$ are there?

Answer

There are exactly $n+1$. We denote them by δ_i^n where $i = 0, \ldots, n$.

$$
\begin{tikzcd}
\mathbf{0} \arrow[r, "\delta_0" description] & \mathbf{1} \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & \mathbf{2} \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=2, "\delta_2"'] & \mathbf{3} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbf{4} & \cdots
\end{tikzcd}
$$

Injectives and surjectives

Question

How many injective monotone $f : \mathbf{n} \to \mathbf{n+1}$ are there?

Answer

There are exactly $n+1$. We denote them by δ_i^n where $i = 0, \ldots, n$.

$$\begin{tikzcd}
\mathbf{0} \arrow[r, "\delta_0" description] & \mathbf{1} \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & \mathbf{2} \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=2, "\delta_2"'] & \mathbf{3} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbf{4} & \cdots
\end{tikzcd}$$

Question

How many surjective monotone $g : \mathbf{n+1} \to \mathbf{n}$ are there?

Injectives and surjectives

Question

How many injective monotone $f : \mathbf{n} \to \mathbf{n+1}$ are there?

Answer

There are exactly $n + 1$. We denote them by δ_i^n where $i = 0, \ldots, n$.

$$\begin{tikzcd}
\mathbf{0} \arrow[r, "\delta_0" description] & \mathbf{1} \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & \mathbf{2} \arrow[r, shift left=2, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=2, "\delta_2"'] & \mathbf{3} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathbf{4} \; \cdots
\end{tikzcd}$$

Question

How many surjective monotone $g : \mathbf{n+1} \to \mathbf{n}$ are there?

Answer

There are exactly n. We denote them by σ_i^n where $i = 0, \ldots, n - 1$.

$$\begin{tikzcd}
\mathbf{1} & \mathbf{2} \arrow[l, "\sigma_0" description] & \mathbf{3} \arrow[l, shift right, "\sigma_0"'] \arrow[l, shift left, "\sigma_1"] & \mathbf{4} \arrow[l, shift right=2, "\sigma_0"'] \arrow[l, "\sigma_1" description] \arrow[l, shift left=2, "\sigma_2"] & \mathbf{5} \; \cdots \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3]
\end{tikzcd}$$

Theorem (coface-codegeneracy decomposition)

Any morphism in $\bar{\Delta}$ can be written as a composition of morphisms δ_i^n, called ***coface maps***, and σ_j^n, called ***codegeneracy maps***. More precisely, an arrow $f : \mathbf{n} \to \mathbf{n}'$ has a unique decomposition as follows:

$$f = \delta_{i_1} \circ \ldots \circ \delta_{i_k} \circ \sigma_{j_1} \circ \ldots \circ \sigma_{j_l}$$

where

$$n' > i_1 > \ldots > i_k \geq 0$$
$$0 \leq j_1 < \ldots < j_l < n - 1$$

and

$$n' = n - l + k$$

Theorem (coface-codegeneracy decomposition)

Any morphism in $\bar{\Delta}$ can be written as a composition of morphisms δ_i^n, called ***coface maps***, and σ_j^n, called ***codegeneracy maps***. More precisely, an arrow $f : \mathbf{n} \to \mathbf{n}'$ has a unique decomposition as follows:

$$f = \delta_{i_1} \circ \ldots \circ \delta_{i_k} \circ \sigma_{j_1} \circ \ldots \circ \sigma_{j_l}$$

where

$$n' > i_1 > \ldots > i_k \geq 0$$
$$0 \leq j_1 < \ldots < j_l < n - 1$$

and

$$n' = n - l + k$$

We can apply this image factorization to measure the size of hom-sets $Hom_\Delta([m],[n])$.

$$|Hom_\Delta([m],[n])| = \sum_k |inj([k],[n])| \times |surj([m],[k])| =$$
$$\sum_k \binom{n+1}{k+1}\binom{m}{k} = \sum_k \binom{n+1}{k+1}\binom{m}{m-k} =$$
$$\binom{n+m+1}{n}$$

The Simplicial Category

Definition

The ***Simplicial category*** Δ is the full subcategory of $\bar{\Delta}$ whose objects are all the positive ordinals. So, the category Δ does not possess the initial object, but it still has the terminal object.

The Simplicial Category

Definition

The ***Simplicial category*** Δ is the full subcategory of $\bar{\Delta}$ whose objects are all the positive ordinals. So, the category Δ does not possess the initial object, but it still has the terminal object.

Remark

In order to comply with the notations of topologists, we use different notation for objects of the category Δ. We denote the object $\mathbf{n+1}$ of $\bar{\Delta}$ by $[n] = \{0, 1, \ldots, n\}$ in Δ for all $n \geq 0$. Standard affine simplex functor $\Delta^{\bullet} \colon \Delta \to Top$ which sends numeral $[n]$ standard affine n-simplex $|\Delta^n|$ establishes an equivalence between the simplicial category and category of affine simplices.

The Simplicial Category

In the simplicial category, for any integer $n \geq 1$, and any $0 \leq i \leq n$ we have $\delta_i^n : [n-1] \to [n]$ as the injective order preserving map skipping i. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $\sigma_j^n : [n+1] \to [n]$ the surjective order preserving map with $(\sigma_j^n)^{-1}(\{j\}) = \{j, j+1\}$

The Simplicial Category

In the simplicial category, for any integer $n \geq 1$, and any $0 \leq i \leq n$ we have $\delta_i^n : [n-1] \to [n]$ as the injective order preserving map skipping i. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $\sigma_j^n : [n+1] \to [n]$ the surjective order preserving map with $(\sigma_j^n)^{-1}(\{j\}) = \{j, j+1\}$

Remark

For each $n \geq 0$, there are exactly $n+1$ coface maps $\delta_i : [n-1] \to [n]$ and $n+1$ codegenracy maps $\sigma_i : [n+1] \to [n]$. These maps, as we have seen already, generate all morphisms in the simplicial category.

The Simplicial Category

Furthermore, for each $n \geq 0$, we have a string of adjunctions:

$$\delta_{n+1} \dashv \sigma_n \dashv \delta_n \dashv \ldots \dashv \sigma_0 \dashv \delta_0$$

where unit of $\delta_{k+1} \dashv \sigma_k$ and counit of $\sigma_k \dashv \delta_k$ are identities. From these one obtains:

$$\delta_{k+1} = \delta_{k+1} \circ \sigma_k \circ \delta_k \leq \delta_k$$

Similarly, one obtains:

$$\sigma_{k-1} = \sigma_k \circ \delta_k \circ \sigma_{k-1} \leq \sigma_k$$

Note: It follows that δ_k is a section of both σ_k and σ_{k+1}. Similarly, δ_k and δ_{k+1} are both sections of σ_k.

$$\begin{tikzcd}
{[n+1]} \arrow[d, shift right=3ex, "\sigma_{k+1}"'] \arrow[d, shift left=3ex, "\sigma_k"] \\
{[n]} \arrow[u, "\delta_k" description]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
{[n+1]} \arrow[d, "\sigma_k" description] \\
{[n]} \arrow[u, shift left=3ex, "\delta_{k+1}"] \arrow[u, shift right=3ex, "\delta_k"']
\end{tikzcd}$$

Cosimplicial identities

Following diagrams commute:

$$\begin{tikzcd}
{[n-1]} \arrow[r, "\delta_i^n"] \arrow[d, "\delta_{j-1}^n"'] & {[n]} \arrow[d, "\delta_j^{n+1}"] \\
{[n]} \arrow[r, "\delta_i^{n+1}"'] & {[n+1]}
\end{tikzcd}$$

when

$0 \leq i < j \leq n+1$

$$\begin{tikzcd}
{[n+1]} \arrow[r, "\sigma_i^n"] \arrow[d, "\sigma_{j+1}^n"'] & {[n]} \arrow[d, "\sigma_j^{n-1}"] \\
{[n]} \arrow[r, "\sigma_i^{n-1}"'] & {[n-1]}
\end{tikzcd}$$

when

$0 \leq i \leq j \leq n-1$

Cosimplicial identities

And also:

$$\begin{tikzcd} {[n-1]} \arrow[r, "\delta_i^n"] \arrow[d, "\sigma_j^{n-2}"'] & {[n]} \arrow[d, "\sigma_j^{n-1}"] \\ {[n-2]} \arrow[r, "\delta_{i-1}^{n-1}"'] & {[n-1]} \end{tikzcd}$$

when

$$0 < j + 1 < i \leq n$$

$$\begin{tikzcd} {[n-1]} \arrow[r, "\delta_i^n"] \arrow[d, "\sigma_{j-1}^{n-2}"'] & {[n]} \arrow[d, "\sigma_j^{n-1}"] \\ {[n-2]} \arrow[r, "\delta_i^{n-1}"'] & {[n-1]} \end{tikzcd}$$

when

$$0 \leq i < j \leq n - 1$$

Cosimplicial identities

And finally,

$$\begin{tikzcd}
{[n-1]} \arrow[r, "\delta_j^n"] \arrow[d, "\delta_{j+1}^n"'] \arrow[dr, "id"] & {[n]} \arrow[d, "\sigma_j^{n-1}"] \\
{[n]} \arrow[r, "\sigma_j^{n-1}"'] & {[n-1]}
\end{tikzcd}$$

when

$$0 \leq j \leq n-1$$

Cosimplicial identities

Theorem

The category Δ is the universal category with objects $[n]$, $n \geq 0$ and coface and codegeneracy morphisms such that:

- *Every morphism is a composition of coface and codegeneracies.*
- *The cosimplicial identities are satisfied and any identity relation among the morphisms is generated by these identities.*

Pushout in simplicial category

For each $n \geq 2$, the object $[n+1]$ can be constructed as the pushout

$$\begin{tikzcd} {[n-1]} \arrow[r, "\delta_0"] \arrow[d, "\delta_{n-1}"'] & {[n]} \arrow[d, "\delta_n"] \\ {[n]} \arrow[r, "\delta_0"'] & {[n+1]} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

What's more, this diagram is in fact a 2-categorical limits, that is a bipushout as was observed in
Ross Street (1980). "Fibrations in bicategories". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 21.2, pp. 111–160

Pushout in simplicial category

Remark

This means that $\bar{\Delta}$ is generated as a 2-category by these pushouts and by taking for each $n \geq 0$, successive adjoints of morphisms in $Hom(\mathbf{n}, \mathbf{n+1})$, starting from δ_{n+1}.

Simplicial objects

Definition

- The presheaf category $Psh(\Delta) = Fun(\Delta^{\mathrm{op}}, \mathrm{Set})$ is called **category of simplicial sets** and its objects are called **simplicial sets**. Traditionally one denotes the category of simplicial sets by $s\mathrm{Set}$.
- Generally, $s\mathcal{A} = Fun(\Delta^{\mathrm{op}}, \mathcal{A})$ denotes the category of simplicial objects in a category $\mathcal{A}$.

Example

- $s^2\mathrm{Set} = s(s\mathrm{Set})$ is the category of bisimplicial sets.
- $sGrp$ is the category of simplicial groups.
- $sRing$ is the category of simplicial rings.
- And so on.

To know more about these categories, see Peter May's *Simplicial objects in algebraic topology , University of Chicago Press, 1967*.

Unwinding the information of a simplicial object

Let $\mathcal{C}$ be a category.

- Given a simplicial object X in $\mathcal{C}$ we obtain a sequence of objects $X_n = X[n]$ endowed with the morphisms $d_j = X(\delta_j) : X_n \to X_{n-1}$ and $s_j = X(\sigma_j) : X_n \to X_{n+1}$. These morphisms satisfy the dual of the relations between cofaces and codegeneracies. So, we call d_i and s_j ***face maps*** and ***degeneracy maps*** respectively.
- Conversely, given a sequence of objects X_n and morphisms d_j, s_j satisfying these relations there exists a unique simplicial object U in $\mathcal{C}$ such that $X_n = X[n]$, $d_j = X(\delta_j)$, and $s_j = X(\sigma_j)$.
- A morphism between simplicial objects X and X' is a natural transformation between them.

Density theorem

Theorem

Every simplicial set is canonically a colimit of the standard simplices (aka representable presheaves). That is:

$$\varinjlim \Delta^n \cong X$$

Proof.

Apply Grothendieck construction to the simplicial category Δ and the category Set and you have:

$$\varinjlim_{x \in \mathbf{el}(X)} \Delta^n \cong X$$

Notice that objects of $el(X)$ are of the form $([m], x)$ where $x \in X_m$ is an m-simplex. A morphism $([m], x) \to ([n], y)$ is a morphism $f : [m] \to [n]$ with $X_f(y) = x$. $\square$

Definition

A morphism $f : [m] \to [n]$ in Δ is called ***inflationary*** if $i \leq f(i)$ for every $0 \leq i \leq m$.

Definition

The subcategory of Δ consist only of inflationary morphisms is denoted by Δ_{infl}. Equivalently, Δ_{infl} is the category generated by coface maps.

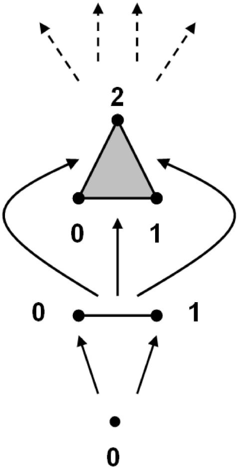


Figure: An illustration of the category Δ_+

A detour to classical algebraic topology

Nomenclature

- The convex hull of a subset $\{v_{i_0}, \ldots, v_{i_k}\}$ of set of points $\{v_0, \ldots, v_n\}$ in an affine space is called a ***k-face*** of $[v_0, \ldots, v_n]$. We denote that by $[v_{i_0}, \ldots, v_{i_k}]$.
- A set K of simplices in $\mathbb{R}^n$ is called a ***simplicial complex*** if the followings hold:
  1. If K contains a simplex then it contains all faces of this complex.
  2. The intersection of two simplices of K is either empty or a common face.
  3. K is locally finite, that is every point of $\mathbb{A}^n$ has a neighbourhood that intersects only finitely many simplices of K. Of course this condition materialises only when K is infinite.

An example of a simplicial complex

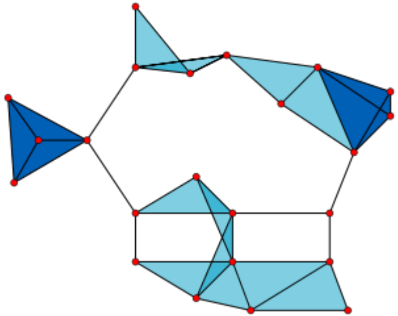


Figure: *An example of a simplicial complex*

A non-example of a simplicial complex

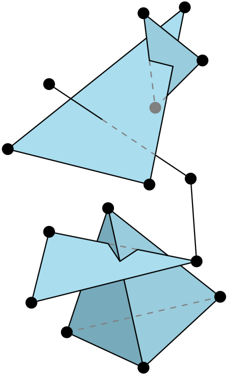
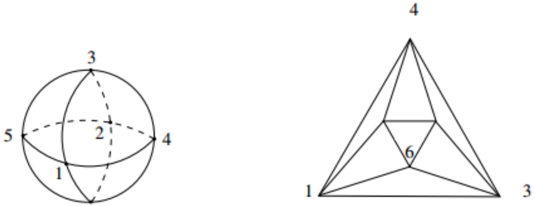
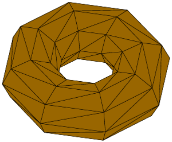


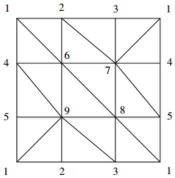
Figure: *A non-example of simplicial complex*

Simplicial decomposition of sphere and torus



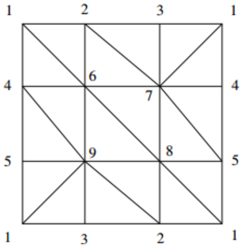
Simplicial complex on a sphere.



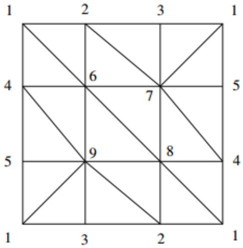


Simplicial complex on a torus.

Simplicial decomposition of Klein bottle and projective plane



Klein bottle



$\mathbb{RP}^2$

Simplicial vs globular

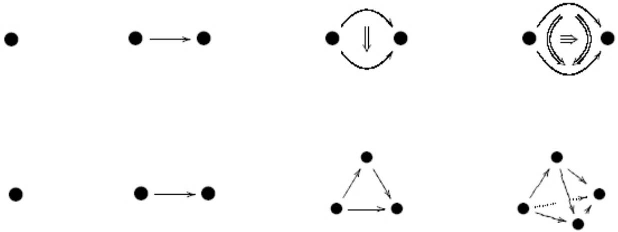


Figure: Globular and simplicial complexes

Standard topological simplex

The functor $\Delta_\bullet : \Delta \to \mathit{Top}$ is defined as follows by its action:

- On objects: it sends $[n]$ to the standard (n-dim) topological simplex

  $$\Delta_n = \{x_0 e_0 + x_1 e_1 + \ldots + x_n e_n | \sum_{0 \leq i \leq n} x_i = 1, x_i \geq 0\} \subseteq \mathbb{R}^{n+1}$$

  where $e_i = (0, 0, \ldots, 1, \ldots, 0, 0)$ with 1 in i-th position and 0 elsewhere.
- On morphisms: it sends $\alpha : [n] \to [m]$ to

  $$\Delta_\alpha(p) = \sum_{0 \leq i \leq n} t_i e_{\alpha(i)}$$

  where $p = t_0 e_0 + t_1 e_1 + \ldots + t_n e_n$ is a point of Δ_n.

$\Delta_\bullet$ in terms of coface and codegeneracy maps

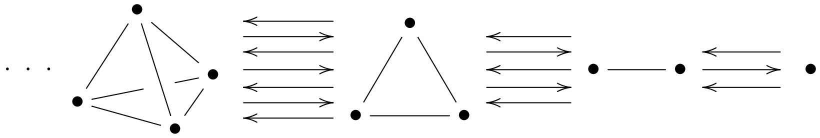
We look at the action of Δ on coface and codegeneracy maps: Let's set $d^i := \Delta_{\delta_i} \colon \Delta_{n-1} \to \Delta_n$ and $s^i := \Delta_{\sigma_i} \colon \Delta_n \to \Delta_{n-1}$.
Then

$$d^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

Also,

$$s^i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, \ldots, t_{n+1})$$

And of course, this data is enough to know the action of the functor on every arrow of category Δ.



(Total) singular complex of a space

Let Y be any topological space. We define a simplicial set SY by setting $SY_n = Top(\Delta_n, Y)$ to be the set of continuous maps from the standard topological n-simplex to the space Y. In algebraic topology, we call elements of SY_n the *n-simplices* of Y.
SY together with the following face and degeneracy maps form a simplicial set which we call ***total singular complex*** of Y.

- Notice that
$$d_i : Top(\Delta_{n+1}, Y) \to Top(\Delta_n, Y)$$
is obtained by postcomposition with $d^i : \Delta_n \to \Delta_{n+1}$. We call $d_i(p)$ the *i-th face* of singular complex p.
- Similarly,
$$s_i : Top(\Delta_{n-1}, Y) \to Top(\Delta_n, Y)$$
is obtained by postcomposition with $s^i : \Delta_n \to \Delta_{n-1}$. We call $s_i(q)$ the *i-th degeneracy* of singular n-simplex.
- A simplex $x \in X_n$ is called *non-degenerate* if x cannot be written as $s_i y$ for any $y \in X_{n-1}$ and any i.

Picturing singular complex functor

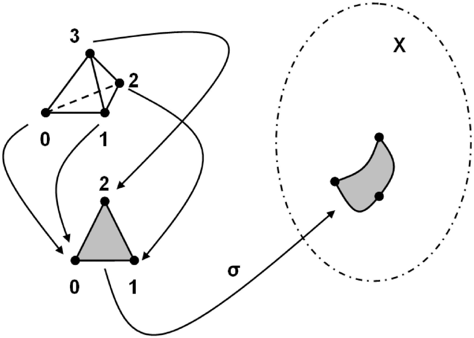


Figure: *Picturing singular complex functor*

Left Kan extension

Let $\mathcal{E}$ be a locally small, cocomplete category and $F : \Delta \to \mathcal{E}$ any functor. We define functor $R : \mathcal{E} \to s\mathbf{Set}$ by

$$(RE)_n = \mathcal{E}(F[n], E)$$

- RE is a simplicial set.
- R admits a left adjoint $L : s\mathbf{Set} \to \mathcal{E}$ which is known as ***left Kan extension*** of F.

$$\begin{tikzcd}
 & s\mathbf{Set} \arrow[dr, dotted, "L"] & \\
\Delta \arrow[ur, "y"] \arrow[rr, "F"'] & & \mathcal{E}
\end{tikzcd}$$

So, we have isomorphism of sets

$$\mathcal{E}(LX, E) \cong s\mathbf{Set}(X, RE)$$

natural in X and E.

Left Kan extension as a coend

The left adjoint L can equivalently be expressed as a coend. That is

$$LX = \int^n X_n \otimes F[n]$$

where

$$X_n \otimes F[n] := \coprod_{X_n} F[n]$$

If $f : [n] \to [m]$ then define f_* as the unique morphism from the coproduct $X_m \otimes F[n]$ of $F[n]$'s to the cocone $X_m \otimes F[m]$: So, the diagram below commutes

$$\begin{tikzcd}
F[n] \arrow[r, "F(f)"] \arrow[d, "x"'] & F[m] \arrow[d, "x"] \\
X_m \otimes F[n] \arrow[r, dashed, "f_*"'] & X_m \otimes F[m]
\end{tikzcd}$$

for all $x \in X_m$.

Left Kan extension as a coend

Also, define f^* as the unique morphism which makes the diagram below commutative for every m-simplex $x \in X_m$.

$$\begin{tikzcd} F[n] \arrow[r, "X_f"] \arrow[d, "x"'] & F[n] \arrow[d, "X_f(x)"] \\ X_m \otimes F[n] \arrow[r, dashed, "f^*"'] & X_n \otimes F[n] \end{tikzcd}$$

LX is the universal wedge:

$$\begin{tikzcd} X_m \otimes F[n] \arrow[r, "f_*"] \arrow[d, "f^*"'] & X_m \otimes F[m] \arrow[d, "\omega_m"] \\ X_n \otimes F[n] \arrow[r, "\omega_n"'] & LX \end{tikzcd}$$

universal wedge

Left Kan extension as a coend

Equivalently LX is given by the coequalizer diagram

$$\coprod_f X_m \otimes F[n] \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{[n]} X_n \otimes F[n] \dashrightarrow LX$$

where the first colimit is taken over all $f : [n] \to [m]$, and $\phi = \coprod f_*$ and $\psi = \coprod f^*$.

Geometric realization

In the case where $\mathcal{E} = Top$ and $F = \Delta_\bullet$ we obtain right adjoint R as total singular complex functor $S : Top \to s\mathrm{Set}$. By construction above it has a left adjoint L.

Definition

The left adjoint to the total singular complex functor $S : Top \to s\mathrm{Set}$ is called the ***geometric realization*** functor and is denoted by $|-| : s\mathbf{Set} \to Top$. It's defined on simplicial sets by

$$|X| = \int^n X_n \times \Delta_n$$

Geometric realization

To know more explicitly what $|-|$ computes for a simplicial set X, we note that $|X|$ is a quotient of the coproduct $\coprod_n X_n \times \Delta_n$ by relations specified in universal wedge. That is a point of $|X|$ is of the form $x \otimes p := [(x, p)]$ where x is an n-simplex for some $n \geq 0$ and $p \in \Delta_n$, and moreover any two points related by some relation generated by following relation are deemed equal in $|X|$:

- $(x, \delta_i(p)) \sim (d_i(x), p)$ for any $x \in X_{n+1}$ and $p \in \Delta_n$.
- $(x, \sigma_i(p)) \sim (s_i(x), p)$ for any $x \in X_{n-1}$ and $p \in \Delta_n$.

Geometric realization

To know more explicitly what $|-|$ computes for a simplicial set X, we note that $|X|$ is a quotient of the coproduct $\coprod_n X_n \times \Delta_n$ by relations specified in universal wedge. That is a point of $|X|$ is of the form $x \otimes p := [(x, p)]$ where x is an n-simplex for some $n \geq 0$ and $p \in \Delta_n$, and moreover any two points related by some relation generated by following relation are deemed equal in $|X|$:

- $(x, \delta_i(p)) \sim (d_i(x), p)$ for any $x \in X_{n+1}$ and $p \in \Delta_n$.
- $(x, \sigma_i(p)) \sim (s_i(x), p)$ for any $x \in X_{n-1}$ and $p \in \Delta_n$.

Example

Suppose X is the trivial simplicial set. That is $X[n] = \{*\}$ and all face and degeneracy maps are identities. then relations by which we quotient out $\coprod_n \Delta_n$ force this coproduct to collapse to a single point. Hence, $|X| = pt$ is a one point space.

Geometric realization in action

Consider following Δ_{infl}-set.

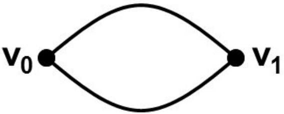


Figure: a Δ_{infl}-set with two vertices and two edges and no degeneracies

Identifying this Δ_{infl}-set with its presheaf representation, we have: $X_0 = \{v_0, v_1\}$ and $X_1 = \{e_0, e_1\}$ with two face maps:

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0$$

We would like to concretely calculate what $|X|$ is as a topological space:

Geometric realization in action

$$|X| = \frac{\coprod_{n \geq 0} X_n \otimes \Delta_n}{\sim} = \frac{\Delta_0 \coprod \Delta_0 \coprod \Delta_1 \coprod \Delta_1}{\sim}$$

So, we have two 0-cells indexed by v_0 and v_1, and two 1-cells indexed by e_0 and e_1. Plus we have the pasting relations:

- $(e_0, \delta_0(*)) \sim (d_0(e_0), *) = (v_1, *)$
- $(e_0, \delta_1(*)) \sim (d_1(e_0), *) = (v_0, *)$
- $(e_1, \delta_0(*)) \sim (d_0(e_1), *) = (v_1, *)$
- $(e_1, \delta_1(*)) \sim (d_1(e_1), *) = (v_0, *)$

So, we get back precisely the shape we started from.

From simplical complexes to simplicial sets and back!

Something more general than our observation in the previous example is true. Start with a simplicial complex X. Construct an associated simplicial set $\bar{X}$ by adjoining all degenerate simplices. We have homeomorphism of spaces:

$$|\bar{X}| \cong X$$

Warning

Notice that, unlike simplicial maps on simplicial complexes, maps on simplicial sets are not completely determined by description of their actions only on vertices (0-simplices). We have to specify their actions across all dimensions.

An example of a Δ_{infl}-set which is not a simplicial complex

Let X be a Δ_{infl}-set with $X_0 = \{v_1, v_2\}$, $X_1 = \{e_1, e_2\}$ and $X_2 = \{*\}$ with

$$X_2 \overset{\mathfrak{d}_0}{\underset{\mathfrak{d}_2}{\xrightarrow{\quad \mathfrak{d}_1 \quad}}} X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0$$

where $\mathfrak{d}_0(*) = e_1$, $\mathfrak{d}_1(*) = e_1$, and $\mathfrak{d}_2(*) = e_2$. Also, $d_0(e_2) = v_1$ and $d_1(e_2) = v_1$. So, e_2 is a loop on v_1. In addition, $d_0(e_1) = v_2$ and $d_1(e_1) = v_1$. The geometric realization of X (depicted below as the quotient space) is *not* a simplicial complex.

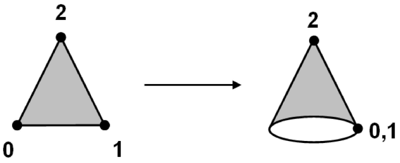


Figure: *An example of a Δ-set*

Going beyond Δ_{infl}-sets: the degeneracy maps

Degeneracy maps are, in some sense, the conceptual converse of face maps. Recall that the face map d_j takes an n-simplex and give us back its jth $(n-1)$-face. On the other hand, the jth degeneracy map s_j takes an n-simplex and gives us back the jth degenerate $(n+1)$-simplex living inside it.

As usual, we illustrate with the standard n-simplex, which will be a model for what happens in all simplicial sets. Given the standard n-simplex $|\Delta^n| = [0, \ldots, n]$, there are $n+1$ *simple* degeneracy maps $s_0, \ldots, s_n$, defined by $s_j[0, \ldots, n] = [0, \ldots, j, j, \ldots, n]$. In other words, $s_j[0, \ldots, n]$ gives us the unique degenerate $n+1$ simplex in $|\Delta^n|$ with only the jth vertex repeated.

Going beyond Δ_{infl}-sets: the degeneracy maps

Example

The standard 0-simplex $X = [0]$, now thought of as a simplicial set, is the unique simplicial set with one element in each X_n, $n \geq 0$. The element in dimension n is $\overbrace{[0, \ldots, 0]}^{n+1 \text{ times}}$.

Example

As a simplicial set, the standard ordered 1-simplex $X = [0, 1]$ already has $n + 2$ elements in each X_n. For example, $X_2 = \{[0, 0, 0], [0, 0, 1], [0, 1, 1], [1, 1, 1]\}$.

Going beyond Δ_{infl}-sets: the degeneracy maps

Picturing simple degeneracies

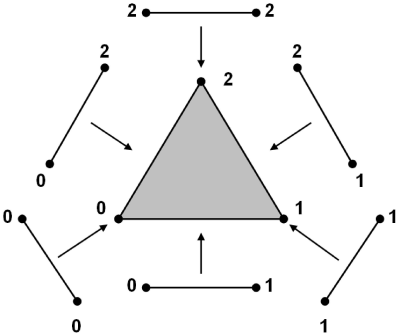


Figure: *simple degenerate simplices of* $|\Delta^2|$

Going beyond Δ_{infl}-sets: the degeneracy maps

Picturing all degeneracies

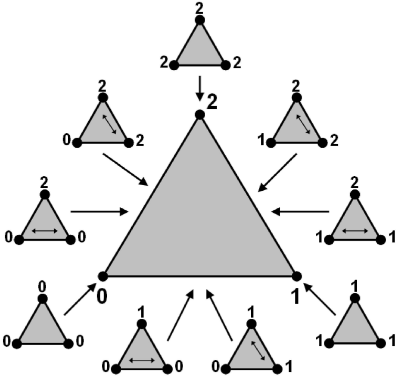


Figure: *all degenerate simplices of* $|\Delta^2|$

Nerve of a category as a simplicial set

Suppose $\mathcal{C}$ is a small category. Define the nerve of $\mathcal{C}$ to be the simplicial set $N\mathcal{C}$ which has:

- $(N\mathcal{C})_0 :=$ vertices of $N\mathcal{C} =$ objects of $\mathcal{C}$
- $(N\mathcal{C})_1 :=$ edges of $N\mathcal{C} =$ arrows of $\mathcal{C}$
- $(N\mathcal{C})_2 :=$ faces of $N\mathcal{C} =$ pairs of composable arrows in $\mathcal{C}$
- $\ldots$

More precisely,

$N\mathcal{C}_n = \{$strings of n composable arrows $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} c_n\}$

Nerve of a category as a simplicial set

The *face maps* $d_i \colon N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ take

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} c_n$$

to

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{i-1}} c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \xrightarrow{f_{i+2}} \ldots \xrightarrow{f_n} c_n$$

for $0 < i < n$ and drops the first (resp. last) arrow for $i = 0$ (resp. $i = n$).

Nerve of a category as a simplicial set

The *face maps* d_i: $N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ take

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$$

to

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i-1}} c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} c_n$$

for $0 < i < n$ and drops the first (resp. last) arrow for $i = 0$ (resp. $i = n$).

And the *degeneracy maps* $s_i : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ send

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$$

to

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_i} c_i \xrightarrow{id} c_i \xrightarrow{f_{i+1}} c_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} c_n$$

Nerve of a category as a simplicial set

Theorem

$N\mathcal{C}_n \cong \mathrm{Map}_{sSet}(\Delta[n], N\mathcal{C})$

Nerve of a category as a simplicial set

Theorem

$N\mathcal{C}_n \cong \mathrm{Map}_{sSet}(\Delta[n], N\mathcal{C})$

Theorem

A category $\mathcal{C}$ can be recovered from its nerve $N\mathcal{C}$ up to isomorphism.

Nerve of a category as a simplicial set

Theorem

$N\mathcal{C}_n \cong \mathrm{Map}_{sSet}(\Delta[n], N\mathcal{C})$

Theorem

A category $\mathcal{C}$ can be recovered from its nerve $N\mathcal{C}$ up to isomorphism.

It would be interesting to see whether we could know exactly which simplicial sets arise as the nerve of a category. What do you think? Is it possible?
We will characterize all of them later when we will have developed enough tools to approach this problem.

Simplicial groups

By definition, a simplicial group G consists of a sequence of groups G_n and collections of group homomorphisms $d_i \colon G_n \to G_{n-1}$ and $s_i \colon G_n \to G_{n+1}$, $0 \leq i \leq n$, that satisfy the face-degeneracy relations.

Remark

Trying to picture group elements as simplices is not as convenient as it was before, since we don't know how to interpret and keep track of extra structures. For example:

- what does it mean geometrically to multiply two simplices?

Simplicial groups

- Let G be a group, and define the category $\mathcal{C}_G$ with exactly one object (let's call it G) and morphisms to be elements of G with composition as multiplication (in the opposite order).
- Nerve $N(\mathcal{C}_G)$ is a simplicial set which can be described as follows:

$$N(\mathcal{C}_G)_n = G^{\times n}$$

the product of G with itself n times. $G^{\times 0}$ is just the trivial group $\{e\}$. For an element $(g_1, \ldots, g_n) \in N(\mathcal{C}_G)_n$:

$$\begin{aligned}
d_0(g_1, \ldots, g_n) &= (g_2, \ldots, g_n) \\
d_i(g_1, \ldots, g_n) &= (g_1, \ldots, g_i g_{i+1}, \ldots g_n) \quad (0 < i < n) \\
d_n(g_1, \ldots, g_n) &= (g_1, \ldots, g_{n-1}) \\
s_i(g_1, \ldots, g_n) &= (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n)
\end{aligned}$$

Simplicial groups

This defines a simplicial group. The geometric realization of $N(\mathcal{C})_G$ is equivalent to the classifying space $\mathbb{B}G$,

$$|N(\mathcal{C}_G)| \simeq \mathbb{B}G$$

and so the simplicial homology $H_*(N(\mathcal{C}_G))$ coincides with group homology of G.

Exercise

1. Show that $|N(\mathcal{C}_G)|$ is the quotient space of a contractible space.
2. Investigate under what criteria imposed on G the classifying space of G is contractible.

$(n-1)$-sphere $\partial\Delta_n$

Definition

The boundary of standard n-simplex (aka representable simplicial set) is defined by:

$$\partial\Delta[n]_m = \{\alpha : [m] \to [n] \mid \alpha \text{ is not surjective}\}$$

which is called $(n-1)$-***sphere*** .

Proposition

$\partial\Delta[n] = \bigcup_{i=0}^{n} \partial^i\Delta[n]$ where

$$\partial^i\Delta[n] = Im(d_i : \Delta[n-1] \to \Delta[n])$$

Note that union is basically a colimit calculated pointwise as is any limit/colimit in a presheaf topos.

Simplicial homotopies

Consider the representable simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \rightrightarrows \Delta[1],$$

coming from the morphisms $[0] \rightrightarrows [1]$ mapping 0 to either elements of $[1] = \{0, 1\}$. Recall also that each set $\Delta[1]_k$ is finite (and has $k + 2$ elements). Hence, if the category $\mathcal{C}$ has finite coproducts, then we can form the simplicial set

$$U \times \Delta[1]$$

for any simplicial object U of $\mathcal{C}$, where

$$(U \times \Delta[1])_k = \coprod_{[k] \to [1]} U_k$$

Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \rightrightarrows U \times \Delta[1].$$

Simplicial homotopies

Definition

Let $\mathcal{C}$ be a category equipped with finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. Let $a, b : U \rightrightarrows V$ be two morphisms.

1. We say a morphism
$$h : U \times \Delta[1] \longrightarrow V$$
is a ***homotopy connecting*** a ***to*** b if $a = h \circ e_0$ and $b = h \circ e_1$.
2. We say morphisms a and b are ***homotopic*** if there exists a homotopy connecting a to b or a homotopy connecting b to a.

Simplicial homotopies

Let $\mathcal{C}$ be a category with finite coproducts. Let U, V be simplicial objects of $\mathcal{C}$. Let $a, b : U \rightrightarrows V$ be morphisms in $\mathcal{C}$. Furthermore, suppose that $h : U \times \Delta[1] \to V$ is a homotopy connecting a to b. For every $n \geq 0$ let us write

$$\Delta[1]_n = \{\alpha_0^n, \ldots, \alpha_{n+1}^n\}$$

where $\alpha_i^n : [n] \to [1]$ for $0 \leq i \leq n+1$ is the morphism

$$\alpha_i^n(j) = \begin{cases} 0 & \text{if} \quad j < i \\ 1 & \text{if} \quad j \geq i \end{cases}$$

Thus

$$h_n : (U \times \Delta[1])_n = \coprod_{\alpha_i^n} U_n \to V_n$$

has a component $h_{n,i} : U_n \to V_n$ which is the restriction to the summand corresponding to α_i^n for all $i = 0, \ldots, n+1$.

Simplicial homotopies

Lemma

In the situation above, we have the following relations:

1. We have $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$.
2. We have $d_j^n \circ h_{n,i} = h_{n-1,i-1} \circ d_j^n$ for $i > j$.
3. We have $d_j^n \circ h_{n,i} = h_{n-1,i} \circ d_j^n$ for $i \leq j$.
4. We have $s_j^n \circ h_{n,i} = h_{n+1,i+1} \circ s_j^n$ for $i > j$.
5. We have $s_j^n \circ h_{n,i} = h_{n+1,i} \circ s_j^n$ for $i \leq j$.

Conversely, a system of maps $h_{n,i}$ satisfying the properties listed above define a homotopy between a and b.

Proof

Left as an exercise for the reader.

Truncation

Let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], [1], [2], \ldots, [n]$. Let $\mathcal{C}$ be a category.

Definition

An *n-**truncated*** simplicial object of $\mathcal{C}$ is a functor $\Delta^{op}_{\leq n} \to \mathcal{C}$. A *morphism of n-truncated simplicial objects* is a natural transformation of functors. We denote the category of n-truncated simplicial objects of $\mathcal{C}$ by the symbol $\mathrm{tr}^n(\mathcal{C})$.

Given a simplicial object U of $\mathcal{C}$ the truncation $\mathrm{tr}^n U$ is the restriction of U to the subcategory $\Delta_{\leq n}$. This defines the *truncation* functor

$$\mathrm{tr}^n : s\mathcal{C} \longrightarrow \mathrm{tr}^n\mathcal{C}$$

from the category of simplicial objects of $\mathcal{C}$ to the category of n-truncated simplicial objects of $\mathcal{C}$.

Kan complexes

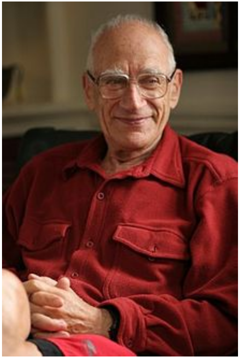


Figure: Daniel Kan (1927-2013)

Kan complexes

Definition

As a simplicial complex, the k-th ***horn*** $|\Lambda^n_k|$ on the n-simplex $|\Delta^n|$ is the subcomplex of $|\Delta^n|$ obtained by removing the interior of $|\Delta^n|$ and the interior of the face $d^k|\Delta^n|$. We take Λ^n_k to refer to the associated simplicial set.

Kan complexes

Definition

As a simplicial complex, the k-th ***horn*** $|\Lambda^n_k|$ on the n-simplex $|\Delta^n|$ is the subcomplex of $|\Delta^n|$ obtained by removing the interior of $|\Delta^n|$ and the interior of the face $d^k|\Delta^n|$. We take Λ^n_k to refer to the associated simplicial set.

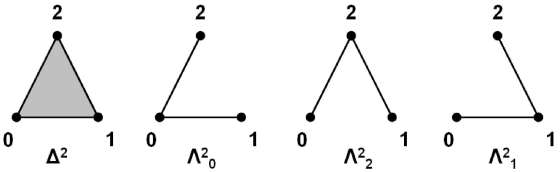


Figure: *The three horns on* $|\Delta^2|$

Kan complexes

Definition

The simplicial object X satisfies the *extension condition* or ***Kan condition*** if any map of simplicial sets $\Lambda^n_k \to X$ can be extended to a simplicial map $\Delta^n \to X$ for all $n \geq 0$ and all $0 \leq k \leq n$. A simplicial set satisfying Kan extension condition is often called a ***Kan complex***.

Example

- The standard simplices Δ^n, $n > 0$ are not Kan complexes.
- However, Δ^0 is a Kan complex.
- $N(\mathcal{C})$ is a Kan complex iff the category $\mathcal{C}$ is a groupoid.
- Any simplicial group is a Kan complex.
  (Thm 2.2. J.C.Moore seminars on algebraic homotopy theory)

Kan complexes

Example

Given a topological space Y, the singular simplicial set $S(Y)$ is a Kan complex. To see this, consider any morphism of simplicial sets $f \colon \Lambda^n_k \to S(Y)$. This is the same as specifying for each $n-1$ face, $d_i\Delta^n$, $i \neq k$, of Δ^n a singular simplex $\sigma_i \colon |\Delta^{n-1}| \to Y$. Every other simplex of Λ^n_k is a face or a degeneracy of a face of one of these $(n-1)$-simplices, and so the rest of the map f is determined by this data.
Furthermore, the compatibility conditions coming from the simplicial set axioms ensure that the topological maps σ_i piece together to yield, collectively, a continuous function $f \colon |\Lambda^n_k| \to Y$. It is left for the reader to confirm this extends to all of $|\Delta^n|$.

Kan complexes

Definition

Two 0-simplices a and b of the simplicial set X are said to be *in the same path component* of X if there is a path p with initial point a and final point b.

Theorem

If X is a Kan complex, then "being in the same path component" is an equivalence relation.

Proof

Not left to the reader; Speaker will give a proof of this!

Definition

$\pi_0 X =$ the set of path components of X.

Topological homotopy theory vs. simplicial homotopy theory

Top. htpy theory	**Simp. htpy theory**
Interval $\mathbb{I}$	Δ^1
Topological space X	Simplicial set X
Continuous maps $X \xrightarrow{f} Y$	Simplicial maps $X \xrightarrow{\theta} Y$
Path: $\mathbb{I} \xrightarrow{p} X$	Path: $\Delta^1 \xrightarrow{p} X$
Initial point: $p(0)$	Initial point: $d_1(p[0,1])$
Final point: $p(1)$	Final point: $d_0(p[0,1])$
...	...

Chain complexes

Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The ***associated chain complex*** $ch_{\mathcal{A}}(U)$ of U, is the chain complex

$$\ldots \to U_2 \to U_1 \to U_0 \to 0 \to 0 \to \ldots$$

with boundary maps $\partial_n : U_n \to U_{n-1}$ given by the formula

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^n .$$

This is a complex because we have

$$\begin{aligned} \partial_n \circ \partial_{n+1} &= (\sum_{i=0}^{n} (-1)^i d_i^n) \circ (\sum_{j=0}^{n+1} (-1)^j d_j^{n+1}) \\ &= 0 \end{aligned}$$

We denote the associated chain complex $s_{\mathcal{A}}(U)$. Clearly, the construction is functorial and hence defines a functor

$$ch_{\mathcal{A}} : s\mathcal{A} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$$

Singular homology functor

The composite

$$Top \xrightarrow{S} s\mathbf{Set} \xrightarrow{F_*} s\mathcal{A}b \xrightarrow{ch} \mathrm{Ch}_{\geq 0} \xrightarrow{H_n} AbGrp$$

where F is free abelian group functor defines a functor which associates to a topological space its n-th ***homology group***. For a space X, its singular homology group is denoted by $H_n(-, \mathbb{Z})$.

Remark

Singular homology groups are conceptually significant invariants of spaces. They assign isomorphic groups to homotopy equivalent spaces. However, it is often hard to calculate them. In most cases, particularly for spaces homeomorphic to some simplicial spaces, we observe that singular homology groups are isomorphic to simplicial homology groups which are much easier to compute.