Kripke-Joyal Semantics for Dependent Type Theory[1]

Pittsburgh's HoTT Seminar

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December 2020

[1] joint work with Steve Awodey (CMU) and Nicola Gambino (Leeds)

Outline

Review of classical Kripke-Joyal semantics for toposes

Classical Kripke-Joyal Semantics for toposes (review) (I)

- **The Kripke–Joyal semantics** of a topos $\mathscr{E}$ gives an interpretation to formulas written in its higher order intuitionistic internal language $HoL(\Sigma_{\mathscr{E}})$.

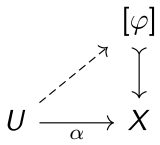
Classical Kripke-Joyal Semantics for toposes (review) (I)

- **The Kripke–Joyal semantics** of a topos $\mathscr{E}$ gives an interpretation to formulas written in its higher order intuitionistic internal language $HoL(\Sigma_{\mathscr{E}})$.
- The Kripke–Joyal semantics is in fact a higher order generalization of the well-known Kripke semantic for intuitionistic propositional logic.

Definition

Let $\mathscr{E}$ be an elementary topos. Given a formula $\varphi(x)$ with a free variable x of sort A in $HoL(\Sigma_{\mathscr{E}})$, and a generalized element $\alpha\colon U \to A$ in $\mathscr{E}$, we define

$$U \Vdash \varphi(\alpha) \quad \Leftrightarrow \quad \alpha \text{ factors through the subobject } [\varphi] \rightarrowtail A.$$



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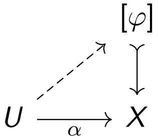

- Call (U, α) the **stage** of forcing.
- Write $\mathscr{E} \Vdash \varphi$ if at every stage (U, α), we have $U \Vdash \varphi(\alpha)$.

Classical Kripke-Joyal semantics for toposes (review) (III)

One can then show:

- $U \Vdash \top(\alpha)$.
- $U \Vdash \bot(\alpha)$ iff U in the initial object of $\mathscr{E}$.
- $U \Vdash (x = x')(\langle \alpha, \alpha' \rangle)$ iff $\alpha \colon U \to X$ and $\alpha' \colon U \to X$ are the same maps in $\mathscr{E}$.
- $U \Vdash (\varphi \wedge \psi)(\alpha)$ iff $U \Vdash \varphi(\alpha)$ and $U \Vdash \psi(\alpha)$.
- $U \Vdash (\varphi \vee \psi)(\alpha)$ iff there are jointly epimorphic arrows $p \colon V \to U$ and $q \colon W \to U$ such that $V \Vdash \varphi(\alpha \circ p)$ and $W \Vdash \varphi(\alpha \circ q)$.
- $U \Vdash (\varphi \Rightarrow \psi)(\alpha)$ iff for any arrow $f \colon V \to U$ such that $V \Vdash \varphi(\alpha \circ f)$ then $V \Vdash \psi(\alpha \circ f)$.
- $c \Vdash \neg\varphi(\alpha)$ iff for all maps $f \colon V \to U$ in $\mathscr{E}$, $V \nVdash \varphi(\alpha.f)$.
- $\vdots$

Classical Kripke-Joyal semantics for presheaf toposes (review) (III)

- Henceforth, $\mathscr{E} = \mathcal{P}\mathsf{Shv}(\mathcal{C}) = \mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$.

Classical Kripke-Joyal semantics for presheaf toposes (review) (III)

- Henceforth, $\mathscr{E} = \mathcal{P}\mathsf{Shv}(\mathcal{C}) = \mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$.
- In the presheaf toposes, every presheaf is a colimit of representables.
- So it is enough to consider forcing statements $U \Vdash \varphi(\alpha)$ for representables $U = \mathrm{y}c$.

Classical Kripke-Joyal semantics for presheaf toposes (review) (III)

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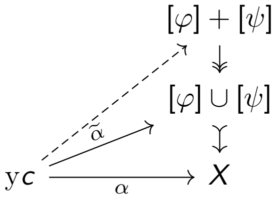
- In the presheaf toposes, every presheaf is a colimit of representables.
- So it is enough to consider forcing statements $U \Vdash \varphi(\alpha)$ for representables $U = \mathrm{y}c$.

$$c \Vdash (\varphi \vee \psi)(\alpha) \Leftrightarrow c \Vdash \varphi(\alpha) \text{ or } c \Vdash \psi(\alpha)$$

Recall $\mathrm{y}c$ is projective & indecomposable.



Limitations of classical Kripke–Joyal semantics

- Bounded quantification. We shall overcome this by generalizing Kripke–Joyal semantics to dependent type theory with universes.
- Equality of terms is extensional and not "up to homotopy'. We will also generalize to homotopy type theory.

Kripke–Joyal semantics for dependent type theory

- Provability versus proof relevance: the classical Kripke–Joyal semantics is only concerned with the provability of a proposition; it is proof irrelevant. Kripke–Joyal semantics for dependent type theory includes the terms (proofs) in the forcing statements.

Kripke–Joyal semantics for dependent type theory

- Provability versus proof relevance: the classical Kripke–Joyal semantics is only concerned with the provability of a proposition; it is proof irrelevant. Kripke–Joyal semantics for dependent type theory includes the terms (proofs) in the forcing statements.
- We want a sound, formal and *(quasi-) mechanical* process to relate internal developments (Cohen et al., 2018), (Orton and Pitts, 2018), etc. with the diagrammatic developments (Gambino and Sattler, 2017), (Sattler, 2017), (Awodey, 2018), etc. found in the models of HoTT literature.

Kripke–Joyal semantics for dependent type theories

Definition (Dependent Kripke–Joyal semantics– forcing terms)

For a context Γ,

Γ

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For a context Γ, a type $\Gamma \vdash A\ \mathtt{Type}$,

$$\begin{array}{c} \Gamma.A \\ \big\downarrow{\scriptstyle p_A} \\ \Gamma \end{array}$$

Definition (Dependent Kripke–Joyal semantics– forcing terms)

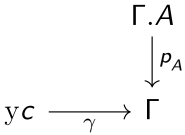
For a context Γ, a type $\Gamma \vdash A$ `Type`, an object c of $\mathcal{C}$,

$$\Gamma.A \xrightarrow{p_A} \Gamma$$

$$\mathrm{y}c \qquad \Gamma$$

Definition (Dependent Kripke–Joyal semantics– forcing terms)

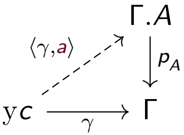
For a context Γ, a type $\Gamma \vdash A$ $\mathtt{Type}$, an object c of $\mathcal{C}$, and a morphism $\gamma \colon \mathrm{y}c \to \Gamma$,



Definition (Dependent Kripke–Joyal semantics– forcing terms)

For a context Γ, a type $\Gamma \vdash A$ Type, an object c of $\mathcal{C}$, and a morphism $\gamma \colon \mathrm{y}c \to \Gamma$, we say c **forces** $a \colon A$ **at stage** γ,

$$c \Vdash [a : A](\gamma) \;\Leftrightarrow\; \text{there is a lift } \langle \gamma, a \rangle \text{ of } \gamma \text{ against } p_A \colon \Gamma.A \to \Gamma.$$



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Proposition

$\Gamma \vdash a : A \;\Leftrightarrow\;$ *There is a family* $(a_\gamma \mid c : \textit{an object of } \mathcal{C}, \gamma \colon \mathrm{y}c \to \Gamma)$ *satisfying*

$$c \Vdash [a_\gamma : A](\gamma)$$

and for every morphism $f \colon d \to c$ *of* $\mathcal{C}$*,*

$$a_\gamma . f = a_{\gamma . f}$$

Proof.

By Yoneda Lemma. □

Forcing dependent sum types

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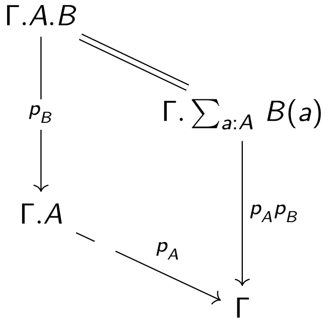
Given a context Γ, *a type* $\Gamma \vdash A \ \mathtt{Type}$, *a type* $\Gamma, x : A \vdash B \ \mathtt{Type}$,



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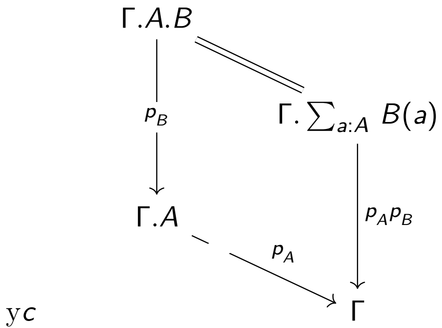

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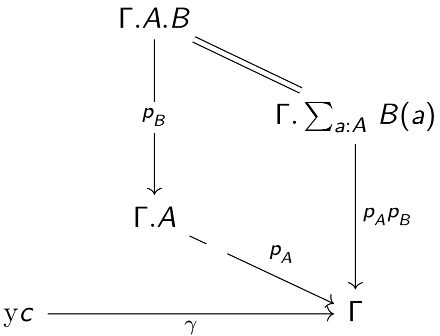
Given a context Γ, *a type* $\Gamma \vdash A$ $\mathtt{Type}$, *a type* $\Gamma, x : A \vdash B$ $\mathtt{Type}$, *an object* c *of* $\mathcal{C}$, *and a morphism* $\gamma : \mathrm{y}c \to \Gamma$,



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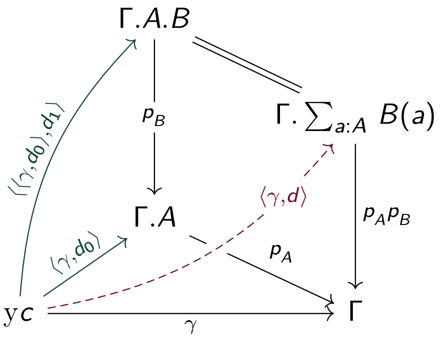
Given a context Γ*, a type* $\Gamma \vdash A$ `Type`*, a type* $\Gamma, x : A \vdash B$ `Type`*, an object* c *of* $\mathcal{C}$*, and a morphism* $\gamma : \mathrm{y}c \to \Gamma$*,*

$$c \Vdash \left[d : \sum_{a:A} B(a)\right](\gamma)$$

iff

$$d = (d_0, d_1)$$
$$c \Vdash [d_0 : A](\gamma)$$
$$c \Vdash [d_1 : B](\langle \gamma, d_0 \rangle) \; .$$



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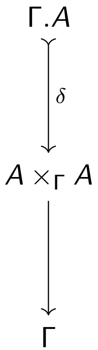

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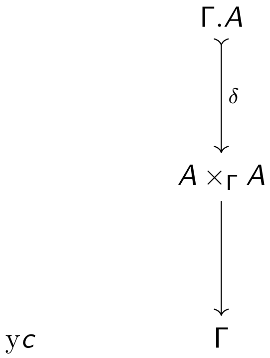

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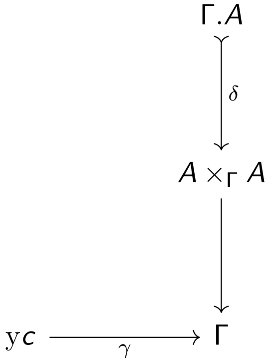

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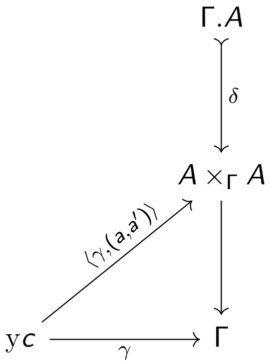
Given a context Γ, a type $\Gamma \vdash A$ $\mathtt{Type}$, an object c of $\mathcal{C}$, a morphism $\gamma \colon \mathrm{y}c \to \Gamma$, $c \Vdash [(a, a') : A \times A](\gamma)$



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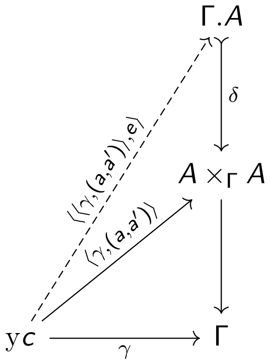
Given a context Γ, *a type* $\Gamma \vdash A$ *Type, an object* c *of* $\mathcal{C}$, *a morphism* $\gamma \colon \mathrm{y}c \to \Gamma$, $c \Vdash [(a, a') : A \times A](\gamma)$ *we have*

$$c \Vdash \left[e : \mathrm{Eq}_A\right](\langle \gamma, (a, a') \rangle) \Leftrightarrow$$

$$a, a' \text{ are equal as morphisms in } \mathscr{E} \Leftrightarrow$$

$$a, a' \text{ are equal elements of } A(c)\ .$$

Type Eq_A is interpreted by the diagonal morphism $\delta \colon A \rightarrowtail A \times_\Gamma A$ over Γ.



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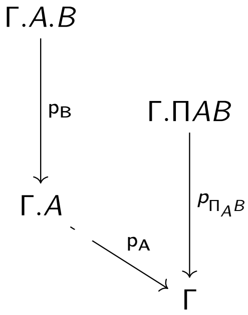

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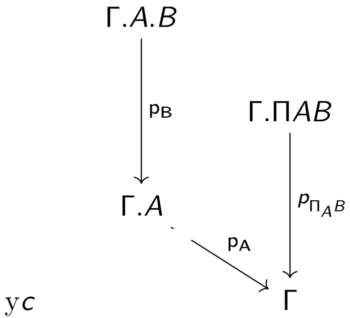

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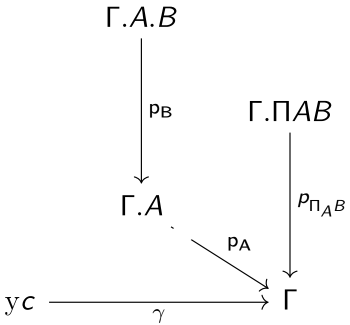

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$$c \Vdash [b : \prod_{x:A} B](\gamma)$$

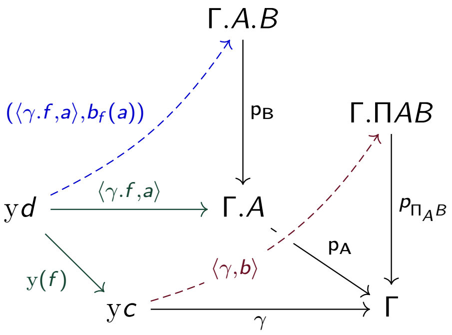
iff for every morphism $f : d \to c$ *in* $\mathcal{C}$,

$$d \Vdash [a : A](\gamma.f)$$

returns

$$d \Vdash [b_f(a) : B](\langle \gamma.f, a \rangle)$$

such that for every morphism $g : d' \to d$, *we have* $b_f(a).g = b_{f \circ g}(a.g)$.



Definition (Dependent Kripke–Joyal semantics– forcing types)

For a type $\Gamma \vdash A\ \mathtt{Type}$, an object c of $\mathcal{C}$, and a morphism $\gamma \colon \mathrm{y}c \to \Gamma$, we say c **forces** A Type **at stage** γ, and we write $c \Vdash [A\,\mathrm{Type}](\gamma)$, whenever there is a presheaf $\widetilde{A}_\gamma$ and a map $p_\gamma \colon \widetilde{A}_\gamma \to \mathrm{y}c$ such that for every morphism $f \colon d \to c$ in $\mathcal{C}$ there is a presheaf $\widetilde{A}_{\gamma.f}$ and a choice of map $\widetilde{A}_{\gamma.f} \to \widetilde{A}_\gamma$, making a pullback square

$$\begin{array}{ccc} \widetilde{A}_{\gamma.f} & \longrightarrow & \widetilde{A}_\gamma \\ {\scriptstyle p_{\gamma.f}}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle p_\gamma} \\ \mathrm{y}d & \xrightarrow[\mathrm{y}f]{} & \mathrm{y}c \end{array} \quad . \tag{1}$$

Forcing a universe $\mathcal{V}$ of small types (I)

- Let κ be an inaccessible cardinal number.
- Call the sets of size strictly less than κ **small**.
- Write $\mathcal{S}\mathsf{et}_\kappa$ for the category of small sets and $\widehat{\mathcal{C}}_\kappa \triangleq [\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathsf{et}_\kappa]$.
- Call a family $p \colon E \to \Gamma$ **small** whenever all the fibres $E(c) \to \Gamma(c)$ are small.

Recall that the **(κ-)universe** $p_{\mathcal{V}} \colon \mathcal{V}_\bullet \to \mathcal{V}$ in $\mathcal{P}\mathsf{Shv}(\mathcal{C})$ is defined as follows:

1. $$\mathcal{V}c \triangleq \mathsf{Ob}[(\mathcal{C}/c)^{\mathrm{op}}, \mathcal{S}\mathsf{et}_\kappa]$$

2. $$\mathcal{V}_\bullet c \triangleq \mathsf{Ob}[(\mathcal{C}/c)^{\mathrm{op}}, (\mathcal{S}\mathsf{et}_\kappa)_\bullet]$$

3. There is a forgetful map $p_{\mathcal{V}} \colon \mathcal{V}_\bullet \to \mathcal{V}$ which takes (A, a) to A.

Forcing a universe (III)

Proposition

For an object c of $\mathcal{C}$,

$$c \Vdash [a : \mathcal{V}] \Leftrightarrow c \Vdash [\mathsf{El}(a)\mathsf{Type}],$$

$$\mathsf{El}(a.f) \equiv \mathsf{El}(a).f \text{ for every } f : d \to c \text{, and}$$

$$\mathsf{El}(a) \to \mathrm{y}c \text{ and } \mathsf{El}(a.f) \to \mathrm{y}d \text{ (for all } f : d \to c) \text{ are small.}$$

Forcing a universe (III)

Proposition

For an object c of $\mathcal{C}$,

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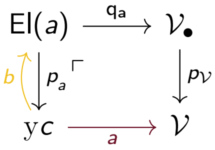
Proposition

For an object c of $\mathcal{C}$,

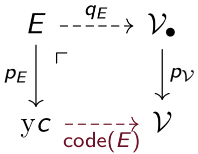
$$c \Vdash [a_\bullet : \mathcal{V}_\bullet] \Leftrightarrow a_\bullet = (a, b) \text{ such that } c \Vdash [a : \mathcal{V}]$$
$$\text{and } c \Vdash [b : \mathrm{El}(a)](\mathrm{id}_c)$$



Forcing universe (IV)

Proposition

For a small family $p_E : E \to \mathrm{y}c$*, we have* $c \Vdash [\mathrm{code}(E) : \mathcal{V}]$ *for a canonical* $\mathrm{code}(E)$.



Remark

- *The "classifying" operation*

$$\mathsf{code}\colon \mathscr{E}_\kappa/\Gamma \to \mathscr{E}(\Gamma, \mathcal{V})$$

has a left (quasi-)inverse, namely the evident "pullback of $p_{\mathcal{V}}\colon \mathcal{V}_\bullet \to \mathcal{V}$*" operation*

$$\mathsf{El}\colon \mathscr{E}(\Gamma, \mathcal{V}) \to \mathscr{E}_\kappa/\Gamma$$

pseudo-naturally in Γ*.*

- *But there is no corresponding uniqueness of classifying maps, relating* $a\colon \Gamma \to \mathcal{V}$ *and* $\mathsf{code}\,\mathsf{El}(a)\colon \Gamma \to \mathcal{V}$*.*
- *We do get the uniqueness of classifying maps by restricting to a smaller universe* Ω *consisting of only "propositions".*

Forcing the impredicative universe Ω of propositions

- As usual, a impredicative universe Ω of (small) propositions in $\mathscr{E}$ is defined object-wise by
$$\Omega(c) \triangleq \mathrm{Ob}\,\mathcal{F}\mathrm{un}((\mathcal{C}/c)^{\mathrm{op}}, \mathbb{2})\ ,$$
where $\mathbb{2}$ is the category with two objects, say $\bot$, $\top$, and one non-identity arrow $\bot \to \top$.
- $\Omega(c)$ is isomorphic to the set of sieves on object c, or equivalently, the set of subobjects of $\mathrm{y}c$.

Forcing the impredicative universe Ω of propositions

$$\Omega(c) \triangleq \mathsf{Ob}\,\mathcal{F}\mathsf{un}((\mathcal{C}/c)^{\mathrm{op}}, \mathbb{2}) ,$$

Proposition

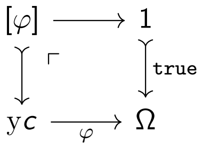
Given an object c of $\mathcal{C}$, the following statements are equivalent:

1. $c \Vdash [\varphi : \Omega]$.
2. $c \Vdash \left[[\varphi]\ \mathsf{Type}\right]$ *such that the maps* $\mathsf{p_c} : [\varphi] \rightarrowtail \mathrm{y}c$ *and* $\mathsf{p_f} : [\varphi.f] \rightarrowtail \mathrm{y}d$ *(for all* $f : d \to c$*) are monomorphisms.*

Forcing the impredicative universe Ω of propositions

Proposition

Suppose $c \Vdash [\varphi : \Omega]$. The following statements are equivalent:

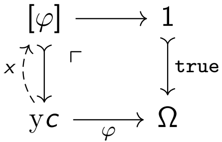


Forcing the impredicative universe Ω of propositions

Proposition

Suppose $c \Vdash [\varphi : \Omega]$. *The following statements are equivalent:*

1. $c \Vdash \left[x : [\varphi]\right]$.
2. $[\varphi] = \mathrm{y}c$.
3. $c \Vdash \left[e : \mathsf{Eq}_{\Omega}(\varphi, \mathtt{true})\right]$ *for some* e.



Propositions as types (I)

There is a canonical map $\iota \colon \Omega \to \mathcal{V}$ which fits into a cartesian square

$$\begin{array}{ccc} 1 & \xrightarrow{\widetilde{*}} & \mathcal{V}_\bullet \\ {\scriptstyle \mathtt{true}}\downarrow & \lrcorner & \downarrow {\scriptstyle p_{\mathcal{V}}} \\ \Omega & \underset{\iota}{\rightarrowtail} & \mathcal{V} \end{array}$$

where $\widetilde{*} = (\iota\mathtt{true}, *)$, and $*$ is the unique term of $\mathsf{El}(\iota\mathtt{true}) \cong [\mathtt{true}] \cong 1$.

Propositions as types (II)

Proposition

Suppose $\Gamma.A \vdash \varphi : \Omega$. *We have*

1. $\iota(\forall x : A.\varphi(x)) \equiv \Pi(a, \iota\varphi)$.
2. $\Sigma(a, \iota\varphi) \to \iota(\exists x : A.\varphi(x))$ *is inhabited.*

Forcing cofibrant propositions (I)

- As in (Orton and Pitts, 2018), we consider a modality $\mathtt{cof} : \Omega \to \Omega$ satisfying:

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  (i) $\texttt{cof} \circ \texttt{true} = \texttt{true}$,

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  (i) $\mathtt{cof} \circ \mathtt{true} = \mathtt{true}$,
  (ii) $\mathtt{cof} \circ \mathtt{false} = \mathtt{true}$,

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- As in (Orton and Pitts, 2018), we consider a modality $\texttt{cof} : \Omega \to \Omega$ satisfying:
  (i) $\texttt{cof} \circ \texttt{true} = \texttt{true}$,
  (ii) $\texttt{cof} \circ \texttt{false} = \texttt{true}$,
  (iii) $\forall(\varphi, \psi : \Omega).\ \texttt{cof}\,\varphi \Rightarrow (\varphi \Rightarrow \texttt{cof}\,\psi) \Rightarrow \texttt{cof}(\varphi \wedge \psi)$.

Forcing cofibrant propositions (I)

- As in (Orton and Pitts, 2018), we consider a modality $\mathtt{cof} : \Omega \to \Omega$ satisfying:
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- The last axiom is called the **principle of dominance**.

- Obtain $m_{\texttt{Cof}} : \texttt{Cof} \rightarrowtail \Omega$ as the comprehension subtype; in the internal language

$$\texttt{Cof} \triangleq \{\varphi \in \Omega \mid \texttt{cof}\ \varphi\}$$

$$\begin{array}{ccc} \texttt{Cof} & \longrightarrow & 1 \\ {\scriptstyle m_{\texttt{cof}}}\downarrow \ulcorner & & \downarrow{\scriptstyle \texttt{true}} \\ \Omega & \xrightarrow[\texttt{cof}]{} & \Omega \end{array}$$

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- $\mathtt{cof}\,(\mathtt{true}) = \mathtt{true}$ implies that $\mathtt{true} = m_{\mathtt{Cof}} \circ \mathtt{t}$ for a monomorphism $\mathtt{t} \colon 1 \rightarrowtail \mathtt{Cof}$.

$$\begin{array}{ccccc} 1 & \overset{\mathtt{t}}{\rightarrowtail} & \mathtt{Cof} & \longrightarrow & 1 \\ \Big\| & \ulcorner & {\scriptstyle m_{\mathtt{Cof}}}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle \mathtt{true}} \\ 1 & \underset{\mathtt{true}}{\rightarrowtail} & \Omega & \xrightarrow[\mathtt{cof}]{} & \Omega \end{array}$$

Forcing cofibrant propositions (II)

- Obtain $m_{\mathtt{Cof}} : \mathtt{Cof} \rightarrowtail \Omega$ as the comprehension subtype; in the internal language

$$\mathtt{Cof} \triangleq \{\varphi \in \Omega \mid \mathtt{cof}\,\varphi\}$$

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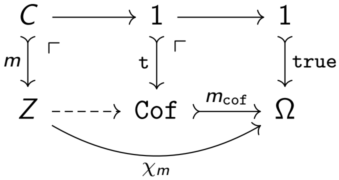
- Call $\mathtt{t}$ the **generic cofibrant proposition**.

Cofibrations

- A monomorphism $m\colon C \rightarrowtail Z$ is a **cofibration** if its classifying map $\chi_m\colon Z \to \Omega$ factors through $m_{\mathtt{cof}}\colon \mathtt{Cof} \rightarrowtail \Omega$.



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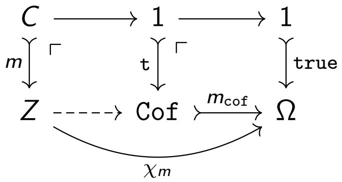

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Cofibrations

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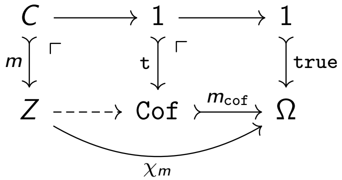

- Therefore, a monomorphism $m \colon C \rightarrowtail Z$ is a cofibration iff it is a pullback of the generic cofibration $\texttt{t} \colon 1 \rightarrowtail \texttt{Cof}$.

Proposition

$m \colon C \rightarrowtail Z$ *is a cofibration* $\Leftrightarrow \mathscr{E} \Vdash \forall z : Z.\, \texttt{cof}(\exists c : C. m(c) = z)$.

Forcing dominance (I)

Consider the following polynomials

$$1 \xrightarrow{\mathtt{t}} \mathtt{Cof} \qquad\qquad \mathscr{E} \xrightarrow{P_{\mathtt{t}}} \mathscr{E}$$

$$1 \xrightarrow{\mathtt{true}} \Omega \qquad \text{associated polynomial functor} \Longrightarrow \qquad \mathscr{E} \xrightarrow{P_{\mathtt{true}}} \mathscr{E}$$

$$\mathcal{V}_\bullet \xrightarrow{p_{\mathcal{V}}} \mathcal{V} \qquad\qquad \mathscr{E} \xrightarrow{P_{p_{\mathcal{V}}}} \mathscr{E}$$

where

$$P_{\mathtt{t}}(A) = \sum_{\varphi : \mathtt{Cof}} A^{[\varphi]}$$

$$P_{\mathtt{true}}(A) = \sum_{\varphi : \Omega} A^{[\varphi]}$$

$$P_{p_{\mathcal{V}}}(A) = \sum_{a : \mathcal{V}} A^{\mathsf{El}(a)}$$

Forcing dominance (II)

Because the square

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ {\scriptstyle \mathtt{t}}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathtt{true}} \\ \mathtt{Cof} & \overset{m_{\mathtt{cof}}}{\rightarrowtail} & \Omega \end{array}$$

is cartesian, we obtain a cartesian square:

$$\begin{array}{ccc} P_{\mathtt{t}}(\mathtt{Cof}) & \rightarrowtail & P_{\mathtt{true}}(\mathtt{Cof}) \\ {\scriptstyle P_{\mathtt{t}}(m_{\mathtt{cof}})}\downarrow & \ulcorner & \downarrow{\scriptstyle P_{\mathtt{true}}(m_{\mathtt{cof}})} \\ P_{\mathtt{t}}(\Omega) & \rightarrowtail & P_{\mathtt{true}}(\Omega) \end{array} \quad = \quad \begin{array}{ccc} \sum_{\varphi : \mathtt{Cof}} \mathtt{Cof}^{[\varphi]} & \rightarrowtail & \sum_{\varphi : \Omega} \mathtt{Cof}^{[\varphi]} \\ \downarrow & \ulcorner & \downarrow \\ \sum_{\varphi : \mathtt{Cof}} \Omega^{[\varphi]} & \rightarrowtail & \sum_{\varphi : \Omega} \Omega^{[\varphi]} \end{array}$$

Forcing dominance (II)

And because

$$
\begin{array}{ccc}
1 & \xrightarrow{\widetilde{*}} & \mathcal{V}_\bullet \\
{\scriptstyle \texttt{true}} \big\downarrow & \ulcorner & \big\downarrow {\scriptstyle p_{\mathcal{V}}} \\
\Omega & \underset{\iota}{\rightarrowtail} & \mathcal{V}
\end{array}
$$

is cartesian, we obtain a cartesian square:

$$
\begin{array}{ccc}
P_{\texttt{true}}(\Omega) & \rightarrowtail & P_{p_{\mathcal{V}}}(\Omega) \\
{\scriptstyle P_{\texttt{true}}(\iota)} \big\downarrow & \ulcorner & \big\downarrow {\scriptstyle P_{p_{\mathcal{V}}}(\iota)} \\
P_{\texttt{true}}(\mathcal{V}) & \rightarrowtail & P_{p_{\mathcal{V}}}(\mathcal{V})
\end{array}
\quad = \quad
\begin{array}{ccc}
\sum_{\varphi : \Omega} \Omega^{[\varphi]} & \rightarrowtail & \sum_{a : \mathcal{V}} \Omega^{\mathsf{El}(a)} \\
\big\downarrow & \ulcorner & \big\downarrow \\
\sum_{\varphi : \Omega} \mathcal{V}^{[\varphi]} & \rightarrowtail & \sum_{a : \mathcal{V}} \mathcal{V}^{\mathsf{El}(a)}
\end{array}
$$

$$
\begin{array}{ccccc}
P_{\mathtt{t}}(\mathtt{Cof}) & \rightarrowtail & P_{\mathtt{true}}(\mathtt{Cof}) & & \\
\downarrow & \ulcorner & \downarrow & & \\
P_{\mathtt{t}}(\Omega) & \rightarrowtail & P_{\mathtt{true}}(\Omega) & \rightarrowtail & P_{p_{\mathcal{V}}}(\Omega) \\
 & & \downarrow & \ulcorner & \downarrow \\
 & & P_{\mathtt{true}}(\mathcal{V}) & \rightarrowtail & P_{p_{\mathcal{V}}}(\mathcal{V})
\end{array}
$$

Therefore, there is a composite map $P_{\mathtt{t}}(\mathtt{Cof}) \rightarrowtail P_{\mathtt{true}}(\Omega) \rightarrowtail P_{p_{\mathcal{V}}}(\mathcal{V})$ which takes (φ, ψ) to $(\iota\varphi, \iota\psi)$.

Proposition

$\mathscr{E} \Vdash [\mathsf{dom} : \forall(\varphi, \psi : \Omega).\ \mathtt{cof}\,\varphi \Rightarrow (\varphi \Rightarrow \mathtt{cof}\,\psi) \Rightarrow \mathtt{cof}(\varphi \wedge \psi)] \Leftrightarrow$ *there is a lift* dom *of* Σ *making the square commute.*

$$
\begin{array}{ccc}
P_{\mathtt{t}}(\mathtt{Cof}) & \overset{\mathsf{dom}}{\dashrightarrow} & \mathtt{Cof} \\
\downarrow & & \downarrow \iota \\
P_{p_{\mathcal{V}}}(\mathcal{V}) & \underset{\Sigma}{\longrightarrow} & \mathcal{V}
\end{array}
$$

- Note – $\Sigma\colon P_{p_{\mathcal{V}}}(\mathcal{V}) \to (\mathcal{V})$ in above is the Natural Model (resp. CwF) interpretation of the $\sum$ type-former following (Awodey, 2018).

Forcing dominance (IV)

Proposition

For $\varphi : \texttt{Cof}$ *and* $\psi : [\varphi] \to \texttt{Cof}$*, the following statements hold:*

(i) $\mathsf{dom}(\texttt{t}, \varphi) = \varphi = \mathsf{dom}(\varphi, \texttt{t})$.

(ii) $\mathsf{dom}(\mathsf{dom}(\varphi, \psi), \theta) = \mathsf{dom}(\varphi, \mathsf{dom}(\psi, \theta))$.

(iii) $[\mathsf{dom}(\varphi, \psi)] \equiv \sum_{x : [\varphi]} [\psi(x)]$.

Proof.

For (i), note that $\iota(\texttt{t}) = \mathsf{code}(\mathbf{1})$ where $\mathbf{1}$ is the terminal type. Since $\sum_{* : \mathbf{1}} \varphi(*) = \iota\varphi$ and ι is monic, $\mathsf{dom}(\texttt{t}, \varphi) = \varphi$.

For (ii), since $\sum_{x : \iota\varphi} \texttt{t} \cong \mathsf{code}(1)$ and the "Frobenius theorem" for the sum types.

For (iii), observe that

$$[\mathsf{dom}(\varphi, \psi)] \equiv \mathsf{El}\iota(\mathsf{dom}(\varphi, \psi)) \equiv \mathsf{El}(\Sigma(\iota\varphi, \iota\psi)) \equiv \sum_{x : [\varphi]} [\psi(x)] .$$

□

Forcing dominance (V)

Proposition

Cofibrations are closed under composition.

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Proof.

It suffices to prove that if $m_\varphi \colon [\varphi] \rightarrowtail \mathrm{y}c$ and $m_\psi \colon [\psi] \rightarrowtail [\varphi]$ are cofibrations then so is their composite.

Forcing dominance (V)

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Cofibrations are closed under composition.

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It suffices to prove that if $m_\varphi : [\varphi] \rightarrowtail \mathrm{y}c$ and $m_\psi : [\psi] \rightarrowtail [\varphi]$ are cofibrations then so is their composite.

$c \Vdash \left[\varphi : \mathtt{Cof}\right]$ and $c \Vdash \left[\psi : [\varphi] \to \mathtt{Cof}\right]$ imply $c \Vdash \left[\mathrm{dom}(\varphi, \psi) : \mathtt{Cof}\right]$:

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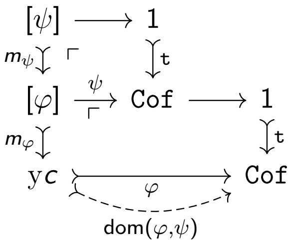
$\mathsf{dom}(\varphi, \psi)$ classifies $m_\varphi \circ m_\psi$ since (i) $[\mathsf{dom}(\varphi, \psi)] \equiv \sum_{x : [\varphi]} [\psi(x)]$, and (ii) $m_\varphi \circ m_\psi$ is the display map of the sum type $\sum_{x : [\varphi]} [\psi(x)]$.



Forcing for partial elements (I)

The **type of partial elements** of a type A is given by the polynomial functor

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Forcing for partial elements (I)

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The type of **cofibrant partial elements** of a type A is given by the polynomial functor

$$A^{+} = P_{\mathtt{t}}(A) = \sum_{\varphi : \mathtt{Cof}} [\varphi] \to A \,.$$

Forcing for partial elements (II)

There is a natural map

$$\begin{aligned} \eta\colon A &\longrightarrow A^{+} \\ a &\longmapsto (\texttt{true}, \lambda * . a : 1 \to A) \end{aligned}$$

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which fits into the pullback square

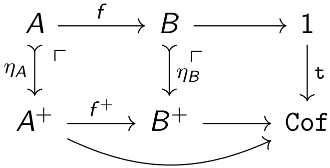
$$\begin{array}{ccc} A & \overset{\eta}{\rightarrowtail} & A^{+} \\ {\scriptstyle !_A}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathtt{fst}} \\ 1 & \underset{\mathtt{t}}{\rightarrowtail} & \mathtt{Cof} \end{array} .$$

Proposition ((Awodey, 2018))

The map $\eta_A \colon A \to A^{+}$ is a cofibration and it classifies partial maps with cofibrant domain.

Forcing for partial elements (III)

In fact, $\eta\colon \mathrm{Id} \Rightarrow +$ is cartesian:



The right square & the outer rectangle are cartesian $\Rightarrow$ The left square is cartesian.

Forcing for partial elements (IV)

Proposition

$c \Vdash [(\varphi, u) : A^+](\gamma) \iff$
$c \Vdash [\varphi : \mathtt{Cof}](\gamma)$ *and for all* $f : d \to c$, *if* $d \Vdash [x : \varphi.f](\gamma.f)$ *then* $d \Vdash [u_f(x) : A](\gamma.f)$,
where $u_f(x).g = u_{fg}(x)$, *for all* $g : d' \to d$.

Proposition

$c \Vdash [(\varphi, u) : A^+](\gamma) \Leftrightarrow$
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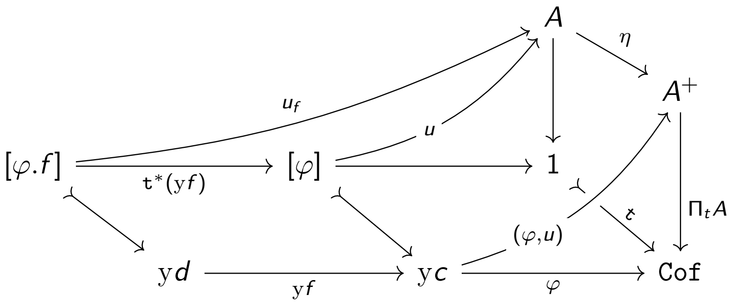

The above gets simplified when $\Gamma = 1$.

$$c \Vdash [(\varphi, u) : A^+] \qquad \Leftrightarrow \qquad \begin{array}{ccccc} 1 & \longleftarrow & [\varphi] & \xrightarrow{u} & A \\ {\scriptstyle \mathtt{t}}\downarrow & \lrcorner & \downarrow{\scriptstyle m} & \ulcorner & \downarrow{\scriptstyle \eta_A} \\ \mathtt{Cof} & \xleftarrow[\varphi]{} & \mathrm{y}c & \xrightarrow[(\varphi,u)]{} & A^+ \end{array}$$

Monad structure from dominance

Proposition ((Awodey, 2018))

$+ \colon \mathscr{E} \to \mathscr{E}$ *is a (fibred) monad.*

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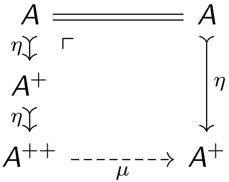
First, we give a category-theoretic proof.

1st Proof.

η_A, η_{A^+}: cofibrations $\Rightarrow$ $\eta_{A^+} \circ \eta_A$: cofibration by dominance.

η_A: cofibrant partial map classifier $\Rightarrow$ there is a unique morphism μ_A classifying the partial map $(\eta_{A^+} \circ \eta_A, \mathrm{id}_A)$.

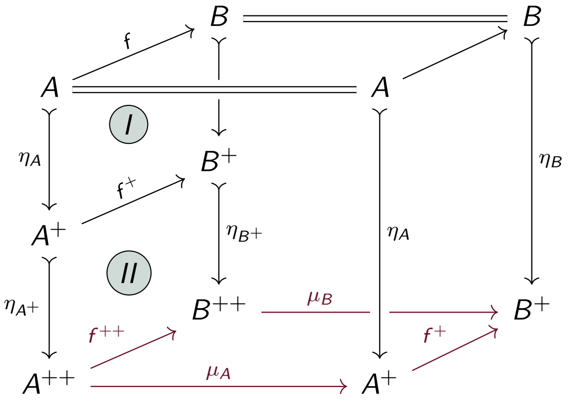


Monad structure from dominance

(1st Proof cont'd.)

μ_A thus obtained is natural in A:

By classifying property of η_B the bottom square commutes since
(i) all vertical squares are pullbacks (I and II because η is cartesian),
(ii) the top square commutes,
(iii) $\eta_{A^+} \circ \eta_A$: cofibration by dominance.



Monad structure from dominance

(1st Proof cont'd.)

To see that $\mu \circ \eta_{A^+} = \mathrm{id}_{A^+}$, observe that the following is a pullback by an easy diagram chase using the previous diagram and the fact that η is always monic.

$$\begin{array}{ccccc}
A & = & = & = & A \\
{\scriptstyle \eta_A}\downarrow & \lrcorner & & & \downarrow{\scriptstyle \eta_A} \\
A^+ & \xrightarrow[\eta_{A^+}]{} & A^{++} & \xrightarrow[\mu]{} & A^+
\end{array}$$

By the uniqueness of the classifying map of (η_A, id_A), we have $\mu_A \circ \eta_{A^+} = \mathrm{id}_{A^+}$. By naturality of η,

$$\eta_{A^+} \circ \eta_A = (\eta_A)^+ \circ \eta_A$$

The same argument above shows

$$\mu_A \circ \eta_{A^+} = \mathrm{id}_{A^+} \quad .$$

Monad structure from dominance (II)

Proposition ((Awodey, 2018))

$+ \colon \mathscr{E} \to \mathscr{E}$ is a (fibred) monad.

Now, we give a proof using Kripke–Joyal semantics.

2nd Proof.

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Now, we give a proof using Kripke–Joyal semantics.

2nd Proof. Write $A^{++} = (A^{+})^{+}$.

$c \Vdash [(\varphi, u) : A^{++}]$

$\Leftrightarrow$ $u = (\psi, u')$, $c \Vdash [\varphi : \mathtt{Cof}]$, and for every $f \colon c' \to c$, if $c' \Vdash [x : \varphi.f]$ then $c' \Vdash [\psi_f(x) : \mathtt{Cof}]$, and for every $g \colon d \to c'$, if $d \Vdash [y : \psi.g]$ then $d \Vdash [u'_g(y) : A]$ and u' is uniform.

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Now, set $f = \mathrm{id}_c$.

The statement above (after $\Leftrightarrow$) becomes $u = (\psi, u')$ and $c \Vdash [\varphi : \mathtt{Cof}]$, $c \Vdash [\psi : [\varphi] \to \mathtt{Cof}]$, $c \Vdash \left[u' : \sum_{x : [\varphi]} [\psi(x)] \to A\right]$

The latter implies

$$c \Vdash [\mathrm{dom}(\varphi, \psi) : \mathtt{Cof}] \text{ and } c \Vdash \big[u' : [\mathrm{dom}(\varphi, \psi)] \to A\big].$$

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Hence

$$c \Vdash \left[(\mathrm{dom}(\varphi, \psi), u') : A^+\right].$$

Monad structure from dominance (II)

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Uniformity of u' implies $\mathscr{E} \Vdash [\mu : A^{++} \to A^{+}]$.

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Uniformity of u' implies $\mathscr{E} \Vdash [\mu : A^{++} \to A^{+}]$.
By Yoneda, we get $\mu \colon A^{++} \to A^{+}$.

Also, $\mu \circ \eta_{A^{+}} = \mathrm{id} = \mu \circ +(\eta_A)$ because $\mathrm{dom}(\varphi, \mathtt{t}) = \varphi$ and $\mathrm{dom}(\mathtt{t}, \psi) = \psi$.
$\mu \circ \mu_{A^{+}} = \mu \circ +(\mu_A)$ because $\mathrm{dom}(\mathrm{dom}(\varphi, \psi), \theta) = \mathrm{dom}(\varphi, \mathrm{dom}(\psi, \theta))$.

For any type A define

$$\mathsf{TFib}(A) := \prod_{\varphi:\mathtt{Cof}} \prod_{u:[\varphi]\to A} \sum_{a:A} u =_{\varphi} a,$$

where the type $u =_{\varphi} a$ (written $(\varphi, u) \nearrow a$ in Orton and Pitts 2018) is defined

$$(u =_{\varphi} a) := \prod_{p:[\varphi]} \mathsf{Eq}_A(up, a)\,.$$

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Proposition

The map $p_A : \Gamma.A \to \Gamma$ is a uniform trivial fibration $\Leftrightarrow$ there is a term $\Gamma \vdash \alpha : \mathrm{TFib}(A)$.

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Recall that p_A being a **a uniform trivial fibration** means that for $p_A : \Gamma.A \to \Gamma$ means that

$$\begin{array}{c} \Gamma.A \\ \Big\downarrow{\scriptstyle p_A} \\ \Gamma \end{array}$$

An application of Kripke–Joyal semantics: Uniform trivial fibration (I)

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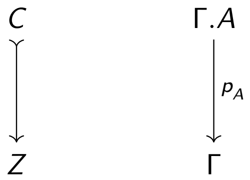

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Recall that p_A being a **a uniform trivial fibration** means that for $p_A : \Gamma.A \to \Gamma$ means that for every cofibration $C \rightarrowtail Z$ and commutative square

$$\begin{array}{ccc} C & \xrightarrow{\quad a \quad} & \Gamma.A \\ \big\downarrow & & \big\downarrow{\scriptstyle p_A} \\ Z & \xrightarrow[\quad z \quad]{} & \Gamma \end{array}$$

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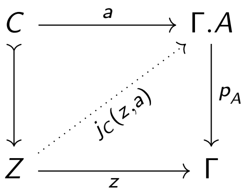
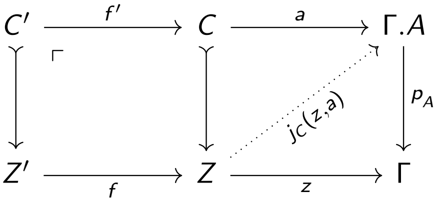
Recall that p_A being a **a uniform trivial fibration** means that for $p_A \colon \Gamma.A \to \Gamma$ means that for every cofibration $C \rightarrowtail Z$ and commutative square there is a diagonal filler $j_C(z,a) \colon Z \to \Gamma.A$ making both triangles commute,



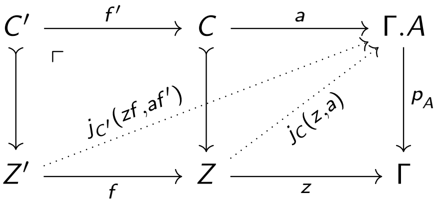
An application of Kripke–Joyal semantics: Uniform trivial fibration (I)

Recall that p_A being a **a uniform trivial fibration** means that for $p_A \colon \Gamma.A \to \Gamma$ means that for every cofibration $C \rightarrowtail Z$ and commutative square there is a diagonal filler $j_C(z,a) \colon Z \to \Gamma.A$ making both triangles commute, and for any map $f \colon Z' \to Z$,



Recall that p_A being a **a uniform trivial fibration** means that for $p_A \colon \Gamma.A \to \Gamma$ means that for every cofibration $C \rightarrowtail Z$ and commutative square there is a diagonal filler $j_C(z, a) \colon Z \to \Gamma.A$ making both triangles commute, and for any map $f \colon Z' \to Z$, these maps j satisfy the indicated condition,

$$j_{C'}(zf, af') = j_C(z, a) \circ f \,.$$



An application of Kripke–Joyal semantics: Uniform trivial fibration (II)

Lemma

For $\Gamma \vdash A\,\mathrm{Type}$, $\gamma : \mathrm{y}c \to \Gamma$ *such that*

$$
\begin{aligned}
&c \Vdash [a : A](\gamma) \\
&c \Vdash [\varphi : \mathtt{Cof}](\gamma) \\
&c \Vdash [u : [\varphi] \to A](\gamma)\,.
\end{aligned}
$$

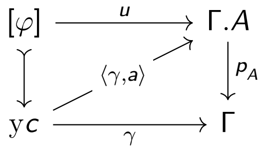
then we also have

$$
c \Vdash \big[e : u =_{\varphi} a\big](\gamma) \qquad \Leftrightarrow \qquad
\begin{array}{ccc}
[\varphi] & \xrightarrow{\quad u \quad} & \Gamma.A \\
\big\downarrow & \overset{\langle \gamma, a\rangle}{\nearrow} & \big\downarrow p_A \\
\mathrm{y}c & \xrightarrow[\quad \gamma \quad]{} & \Gamma
\end{array}
\quad \textit{commutes,}
$$

where

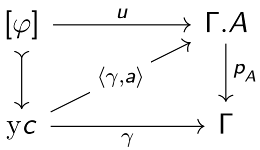
$$
(u =_{\varphi} a) := \prod_{x : [\varphi]} \mathrm{Eq}_A(ux, a)\,.
$$

Proof of Lemma.



$c \Vdash [a : A](\gamma) \Leftrightarrow$ the lower triangle commutes.

Proof of Lemma.

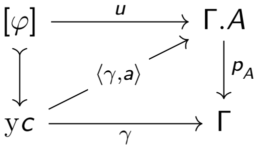


$c \Vdash [a : A](\gamma) \;\Leftrightarrow\;$ the lower triangle commutes.

$c \Vdash [\varphi : \texttt{Cof}](\gamma)$ and $c \Vdash [u : [\varphi] \to A](\gamma) \;\Leftrightarrow\;$ the outer square commutes.

An application of Kripke–Joyal semantics: Uniform trivial fibration (II)

Proof of Lemma.



$c \Vdash [a : A](\gamma) \;\Leftrightarrow\;$ the lower triangle commutes.

$c \Vdash [\varphi : \mathtt{Cof}](\gamma)$ and $c \Vdash [u : [\varphi] \to A](\gamma) \;\Leftrightarrow\;$ the outer square commutes.

$c \Vdash \big[e : u =_\varphi a\big](\gamma)$
$\Leftrightarrow\; c \Vdash \big[e : \prod_{x:[\varphi]} \mathrm{Eq}_A(ux, a)\big](\gamma)$
$\Leftrightarrow$ for all $f : d \to c$ in $\mathcal{C}$, $d \Vdash [x : [\varphi]](\gamma.f)$ returns
$d \Vdash [e_f(x) : \mathrm{Eq}_A(ux, a)]\langle \gamma.f, u[\gamma.f]x, a.f\rangle$
$\Leftrightarrow$ the top triangle commutes. QED.

Proof of Theorem.

Suppose $\Gamma \vdash \alpha : \mathsf{TFib}(A)$.

Thus for all $\gamma \colon \mathrm{y}c \to \Gamma$, we have $c \Vdash [\alpha_\gamma : \mathsf{TFib}(A)](\gamma)$, coherently in γ.

Proof of Theorem.

Suppose $\Gamma \vdash \alpha : \mathsf{TFib}(A)$.

Thus for all $\gamma\colon \mathrm{y}c \to \Gamma$, we have $c \Vdash [\alpha_\gamma : \mathsf{TFib}(A)](\gamma)$, coherently in γ.

Note that

$$\begin{aligned}\mathsf{TFib}(A) &= \prod_{\varphi:\mathtt{Cof}} \prod_{u:[\varphi]\to A} \sum_{a:A} \prod_{x:[\varphi]} \mathsf{Eq}_A(ux, a) \\ &= \prod_{(\varphi,u) : A^+} \sum_{a:A} u =_\varphi a\end{aligned}$$

We thus obtain

$$c \Vdash \left[\alpha_\gamma : \prod_{(\varphi,u) : A^+} \sum_{a:A} u =_\varphi a\right](\gamma) .$$

An application of Kripke–Joyal semantics: Uniform trivial fibration (III)

Proof of Theorem (cont'd).

By Kripke–Joyal semantics of $\prod$ and $\sum$, we have for every $f : d \to c$ in $\mathcal{C}$, if

$$d \Vdash [(\varphi, u) : A^{+}](\gamma.f) \tag{2}$$

then

$$d \Vdash \left[\alpha_{\gamma.f}(\varphi, u)^{0} : A\right](\gamma.f) \tag{3}$$

and

$$d \Vdash \left[\alpha_{\gamma.f}(\varphi, u)^{1} : \left(u =_{\varphi} \alpha_{\gamma.f}(\varphi, u)^{0}\right)\right](\gamma.f) \tag{4}$$

and, for any $g : d' \to d$,

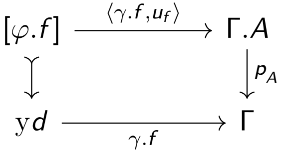
$$\alpha_{\gamma.f}(\varphi, u).g = \alpha_{(\gamma.fg)}(\varphi[g], u[g])\,. \tag{5}$$

An application of Kripke–Joyal semantics: Uniform trivial fibration (III)

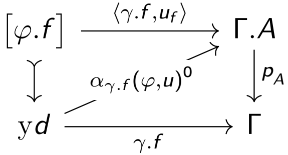
Unfolding the condition (2) yields the following commutative diagram.

$$\begin{array}{ccc}
[\varphi.f] & \xrightarrow{\langle \gamma.f, u_f \rangle} & \Gamma.A \\
\downarrow & & \downarrow p_A \\
\mathrm{y}d & \xrightarrow[\gamma.f]{} & \Gamma
\end{array}$$

Unfolding the condition (2) yields the following commutative diagram.



Lemma applied to (3) and (4) yields the following commuting diagram.

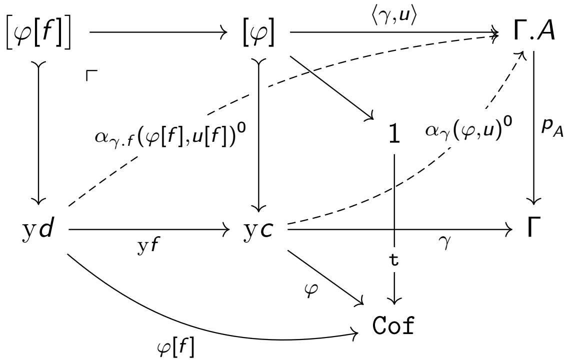


An application of Kripke–Joyal semantics: Uniform trivial fibration (III)

Thus forcing TFib(A) produces diagonal fillers

$$j_\varphi(\gamma, u) \triangleq \alpha_{\gamma.f}(\varphi, u)^0$$

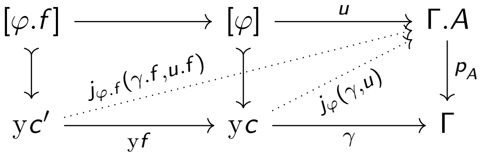
for each lifting problem as in the right hand square below:



An application of Kripke–Joyal semantics: Uniform trivial fibration (III)

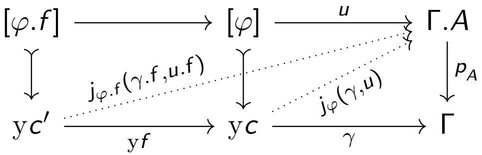
Proof of Theorem (cont'd) – converse argument

If $p_A \colon \Gamma.A \to \Gamma$ is a uniform trivial fibration then in particular for every *basic* cofibration $[\varphi] \rightarrowtail \mathrm{y}c$ and square as on the right below, there is a diagonal filler $j_\varphi(\gamma, u)$ as indicated.

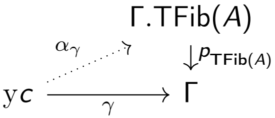


Proof of Theorem (cont'd) – converse argument

If $p_A : \Gamma.A \to \Gamma$ is a uniform trivial fibration then in particular for every *basic* cofibration $[\varphi] \rightarrowtail \mathrm{y}c$ and square as on the right below, there is a diagonal filler $j_\varphi(\gamma, u)$ as indicated.



By the lemma, this corresponds to an element $\alpha_\gamma : \mathrm{y}c \to \mathsf{TFib}(A)$ over $\gamma : \mathrm{y}c \to \Gamma$,



Proof of Theorem (cont'd) – converse argument

The uniformity condition says exactly that for all $f : c' \to c$, the elements α_γ cohere,
$\alpha_{(\gamma.\mathrm{y}f)} = \alpha_\gamma \circ f$.

An application of Kripke–Joyal semantics: Uniform trivial fibration (III)

Proof of Theorem (cont'd) – converse argument

The uniformity condition says exactly that for all $f : c' \to c$, the elements α_γ cohere,
$\alpha_{(\gamma.\mathrm{y}f)} = \alpha_\gamma \circ f$.
By Yoneda for the slice category $\mathscr{E}/\Gamma$ that there is a term $\Gamma \vdash \alpha : \mathsf{TFib}(A)$. QED.

Next ...

Further use of Kripke–Joyal semantics for dependent type theory in

- Extending to uniform **fibrations** using an interval $\mathbb{I}$.
- Showing the fibrancy of path types.
- Showing the universe of fibrations is itself fibrant.
- Showing Frobenius property of fibrations.

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The End

Thanks for your attention!