Bicategories with Base Change

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1 Introduction

The notion of double category was first introduced and studied by Ehresmann in (Ehresmann 1963) and (Ehresmann & Ehresmann 1978). The structure of a weak double category includes a bicategory of horizontal morphisms and a category of vertical morphisms. So, one can view double categories as a generalization of bicategories in which there are two types of morphisms: vertical morphisms that may be taken to compose strictly among themselves and horizontal morphisms that behave like bimodules. Double categories are also useful for making sense of certain constructions such as calculus of mates, spans, and Parametrized Spectra in homotopy theory.

In this notes I will review fibrant double categories (aka framed bicategories introduced in (Shulman 2009)) in which one has a structure of a bicategory and the base change along vertical morphisms. I will give important example of fibrant double category of bimodules of algebras in which base change corresponds to extension, restriction , and coextension of bi-modules along algebra maps.

The idea of a bicategory with base change was first (to my knowledge) formalized in Dominic Veritys PhD thesis (Verity 1992). There he uses concept of 2-categories with proarrow equipment which turns out to be equivalent to framed bicategories.

2 Hom-Tensor adjunction

We have the following bijection of sets

$$\mathbf{Set}(X \times Z, Y) \cong \mathbf{Set}(X, Y^Z)$$

natural in sets X, Y, Z. Note however, that $(-)^Z : \mathbf{Set} \to \mathbf{Set}$ is not a geometric morphism since $(-) \times Z$ is not lex. Similar to the adjunction $(-) \times Z \dashv (-)^Z$ there is one for modules

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of rings:

$$\mathbf{Mod}_R(X \otimes_S Z, Y) \cong \mathbf{Mod}_S(X, \underline{\mathrm{Hom}}_R(Z, Y))$$

where R, S are rings, X is a right S-module, Y is right R-module, Z is a left S-module and a right R-module. $\underline{\mathrm{Hom}}_R(Z, Y)$ is considered as the (left) S-module of R-maps $Z \to Y$. The isomorphism above is natural in X,Y, Z.

Passing to internal constructions in any **Ab**-like monoidal category, we can generalize both of above situations. For this generalization we bear the following principle in mind:

Principle. Since a ring is nothing but a monoid in the monoidal category **Ab** of abelian groups, a internal bimodule **Ab** is a bimodule of rings in usual sense.

By an **Ab**-like monoidal category, I mean a closed monoidal category with equalizers and coequalizers which are stable under tensoring. Suppose that $(\mathcal{V}, \otimes, I)$ is an **Ab**-like monoidal category. Suppose (A, μ, ε) is an internal monoid in there. Define an *internal left A-module* to be the structure (M, m) where M is an object of $\mathcal{V}$ and $m\colon A \otimes M \to M$ is a morphism in $\mathcal{V}$. We refer to m as action map. Moreover, they satisfy unit and associativity axioms. One can form a category $\mathbf{Mod}(\mathcal{V})$ of internal (left) modules in $\mathcal{V}$ in which objects are pairs (A, M), whereby A is a monoid in $\mathcal{V}$, and M is an A-module. Morphisms are pairs (f, ϕ) whereby $f\colon A \to B$ is a monoid morphism and $\phi\colon M \to N$ in $\mathcal{V}$ is f-equivariant, that is the diagram below commutes:

$$\begin{tikzcd}
A \otimes M \arrow[r, "f\otimes\phi"] \arrow[d, "m"'] & B \otimes N \arrow[d, "n"] \\
M \arrow[r, "\phi"'] & N
\end{tikzcd}$$

Identities and composition are identities and composition in $\mathcal{V}$. In fact, there is a Grothendieck fibration of categories

$$\begin{tikzcd}
\mathbf{Mod}(\mathcal{V}) \arrow[d] \\
\mathbf{Mon}(\mathcal{V})
\end{tikzcd}$$

which takes a (left) module (A, M) to its underlying monoid A. The fibre over monoid A is the category A-$\mathbf{Mod}(\mathcal{V})$ of all (left) A-modules. Similarly, one can define notions of internal right module[1] and internal bimodule[2] along the same lines. An (A, B)-bimodule M is given by action map $m\colon A \otimes M \otimes B \to M$ plus expected coherence axioms. Every such bimodule gives rise to a left A-module and a right B-module which can be seen in the diagram below:

$$\begin{tikzcd}
A \otimes M \otimes I \arrow[r, "A\otimes M\otimes\varepsilon_B"] \arrow[d, "\cong"'] & A \otimes M \otimes B \arrow[d, "m"] & I \otimes M \otimes B \arrow[l, "\varepsilon_A\otimes M\otimes B"'] \arrow[d, "\cong"] \\
A \otimes M \arrow[r, dashed, "m_A"'] & M & M \otimes B \arrow[l, dashed, "m_B"]
\end{tikzcd}$$

[1]We use $\mathbf{Mod}_A(\mathcal{V})$ to denote the category of $\mathcal{V}$-internal right A-modules.

[2]We use $\mathbf{Bimod}(\mathcal{V})$ to denote the category of internal $\mathcal{V}$-bimodules.

Suppose M is an (A,B)-bimodule and N is a (B,C)-bimodule. We define tensor product of M and N as the following coequalizer:

$$\begin{tikzcd} M\otimes B\otimes N \arrow[r, shift left, "m_B\otimes 1_N"] \arrow[r, shift right, "1_M\otimes n_B"'] & M\otimes N \arrow[r, dashed, two heads, "q"] & M\otimes_B N \end{tikzcd}$$

The universal property of of q is the familiar universal property of tensor of bi-modules: any bilinear map out of $M\otimes N$ factors via quotient map to $M\otimes_B N$. We now prove that $M\otimes_B N$ is indeed an (A,C)-bimodule. In the diagram below, notice that the top row is again a coequalizer and since both left squares commute, we obtain a unique map $m_A\otimes_B n_C$ between coequalizers which gives $M\otimes_B N$ structure of (A,C)-bimodule.

$$\begin{tikzcd} A\otimes M\otimes B\otimes N\otimes C \arrow[r, shift left, "1_A\otimes m_B\otimes 1_N\otimes 1_C"] \arrow[r, shift right, "1_A\otimes 1_M\otimes n_B\otimes 1_C"'] \arrow[d, "m_A\otimes 1\otimes n_C"'] & A\otimes M\otimes N\otimes C \arrow[r, two heads, "1_A\otimes q\otimes 1_C"] \arrow[d, "m_A\otimes n_C"'] & A\otimes (M\otimes_B N)\otimes C \arrow[d, dashed, "m_A\otimes_B n_C"] \\ M\otimes B\otimes N \arrow[r, shift left, "m_B\otimes 1_N"] \arrow[r, shift right, "1_M\otimes n_B"'] & M\otimes N \arrow[r, two heads, "q"] & M\otimes_B N \end{tikzcd}$$

One may readily show that the operation of tensoring bimodules is associative: that is we have a natural isomorphism of bimodules $(M\otimes_B N)\otimes_C P\cong M\otimes_B(N\otimes_C P)$ whenever either expression is well-formed.

Now, we will generalize construction of internal hom of bimodules from **Ab** to any **Ab**-like monoidal category. Let M be an (A,B)-bimodule, N a (B,C)-bimodule and P a right C-module. Define internal object of C-linear maps as the following equalizer in $\mathcal{V}$:

$$\begin{tikzcd} \underline{\mathrm{Hom}}_C(N,P) \arrow[r, dashed, tail, "e"] & {[N,P]} \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & {[N\otimes C,P]} \end{tikzcd}$$

where ∂_0 and ∂_1 are morphisms in $\mathcal{V}$ whose transpose are given by

$$\begin{tikzcd} {[N,P]\otimes N\otimes C} \arrow[rr, "\widehat{\partial_0}"] \arrow[dr, "\mathsf{ev}\otimes 1_C"'] & & P \\ & P\otimes C \arrow[ur, "p_C"'] & \end{tikzcd} \quad , \quad \begin{tikzcd} {[N,P]\otimes N\otimes C} \arrow[rr, "\widehat{\partial_1}"] \arrow[dr, "1\otimes n_C"'] & & P \\ & {[N,P]\otimes N} \arrow[ur, "\mathsf{ev}"'] & \end{tikzcd}$$

we define a right B-action on $\underline{\mathrm{Hom}}_C(N,P)$ which makes it into a right B-module. First observe that $[N,P]$ is a right B-module with action map $\alpha\colon [N,P]\otimes B\to[N,P]$ with $\widehat{\alpha}=\mathsf{ev}\circ(1_{[N,P]}\otimes n_B)$. Similarly, $[N\otimes C,P]$ is a right B-module with action map $\beta\colon [N\otimes C,P]\otimes B\to[N\otimes C,P]$ with $\widehat{\beta}=\mathsf{ev}\circ(1_{[N,P]}\otimes n_B\otimes 1_C)$. Indeed, by our assumption, operation of tensoring preserves equalizers which implies that both rows of the diagram below are equalizer diagrams and hence there exists a unique morphism $\bar{\alpha}\colon \underline{\mathrm{Hom}}_C(N,P)\otimes B\to$

$\underline{\mathrm{Hom}}_C(N,P)$ which makes the left square commute:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_C(N,P)\otimes B \arrow[r, tail, "e\otimes 1_B"] \arrow[d, dashed, "\bar{\alpha}"'] & {[N,P]\otimes B} \arrow[r, shift left, "\partial_0\otimes 1_B"] \arrow[r, shift right, "\partial_1\otimes 1_B"'] \arrow[d, "\alpha"] & {[N\otimes C,P]\otimes B} \arrow[d, "\beta"] \\
\underline{\mathrm{Hom}}_C(N,P) \arrow[r, tail, "e"] & {[N,P]} \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & {[N\otimes C,P]}
\end{tikzcd}$$

When $\mathcal{V}=\mathbf{Ab}$, $\bar{\alpha}\ (f,b)\ n = f(b\centerdot n)$. $\bar{\alpha}$ gives $\underline{\mathrm{Hom}}_C(N,P)$ structure of right B-module. Moreover, one can prove

$$\mathbf{Mod}_C(\mathcal{V})(M\otimes_B N, P)\cong \mathbf{Mod}_B(\mathcal{V})(M,\underline{\mathrm{Hom}}_C(N,P))$$

natural in A,B,C which establishes internal Hom-tensor adjunction $-\otimes_B N \dashv \underline{\mathrm{Hom}}_C(N,-)$.

Remark *2.1.* For a symmetric monoidal category $(\mathcal{V},\otimes,I,\sigma)$, we can define the notion of internal commutative monoid in $\mathcal{V}$. An internal monoid (M,μ,ε) is commutative whenever

$$\begin{tikzcd}
M\otimes M \arrow[rr, "\sigma_{M,M}"] \arrow[dr, "\mu"'] & & M\otimes M \arrow[dl, "\mu"] \\
& M &
\end{tikzcd}$$

commutes in $\mathcal{V}$. We define the category $\mathbf{CMod}(\mathcal{V})$ whose objects are are bimodules of commutative monoids: $\mathbf{CMod}(\mathcal{V})$ is a sub-category of $\mathbf{Mod}(\mathcal{V})$. A monoid internal in category $\mathbf{CMod}(\mathcal{V})$ is an algebra[3] in $\mathcal{V}$. For instance, $\mathbf{Mon}(\mathbf{CMod}(\mathbf{Ab}))$ is the category of algebra over commutative rings.

Principle. A monoid is a category with one object. A category is oidification of a monoid.

Considering this principle, we can generalize the situation of internal monoids and internal bimodules to internal categories and internal distributors (aka profunctors). However before doing that we will introduce a new formal gadget called weak double category which gives the right framework to help us pursue in that direction. In the next two sections we will introduce strict and weak double categories.

3 Strict double categories

[3] Recall than an algebra over a commutative ring R is an R-module A equipped with R-linear maps $p\colon A\otimes A\to A$ and $i\colon R\to A$ satisfying the associative and unit laws.

Reading list for this section

- First and second sections (Shulman 2009) is a good and easy read on double categories.
- Introductory slides by Thomas Fiore: (Fiore 2007*a*)
- On weak double categories you can read a more substantial paper: Fiore (2007*b*)
- (Pare 2011) is also a good source; although be careful that his notion of weak double category is different than us as he considers composition (only) in vertical direction to be weakly associative and we consider composition (only) in horizontal direction to be weakly associative. The authors develope Yoneda embedding and Yoneda lemma for weak double categories which generalizes Yoneda lemma for bicategories.
- Simona Paoli's interesting book also has some introduction to double categories and more generally n-fold categories. They are investigated as a model of Segal type higher category which provides a proper setting for abstract homotopy theory.

Definition *3.1.* A ***double category*** is an internal category in the (small) category $\mathcal{C}at$ of small categories and functors. Therefore, a double category $\mathbb{D} = (\mathcal{D}_1, \mathcal{D}_0)$ consists of a (small) 'category of objects' $\mathcal{D}_1$ and a (small) 'category of arrows' $\mathcal{D}_0$, with structure functors

$$\begin{aligned} U &\colon \mathcal{D}_1 \to \mathcal{D}_0 \\ L, R &\colon \mathcal{D}_1 \rightrightarrows \mathcal{D}_0 \\ \odot &\colon \mathcal{D}_1 \; {}_R\!\times_L \mathcal{D}_1 \to \mathcal{D}_1 \end{aligned}$$

such that they satisfy usual axioms of an internal category, e.g.

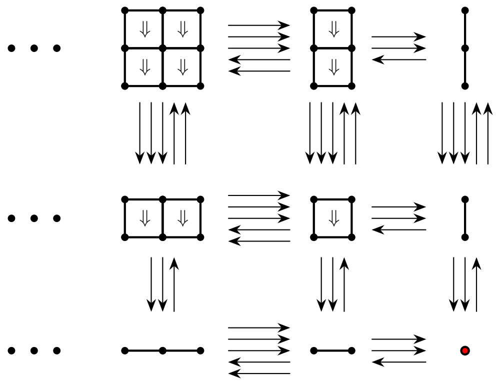
$$\begin{aligned} L(U_A) &= A \\ R(U_A) &= A \\ L(M \odot N) &= LM \\ R(M \odot N) &= RM, \quad \textit{etc.} \end{aligned}$$

Remark *3.2.* If we unwind the above definition, we observe that a double category consists of following sets:

- D_{00}: the set of objects of $\mathcal{D}_0$ = to be called objects of $\mathbb{D}$
- D_{01}: the set of morphisms of $\mathcal{D}_0$ = to be called vertical arrows of $\mathbb{D}$
- D_{02}: the set of composable vertical arrows

- D_{10}: the set of objects of $\mathcal{D}_1$ = to be called horizontal 1-cells of $\mathbb{D}$
- D_{11}: the set of morphisms of $\mathcal{D}_1$ = to be called 2-cells of $\mathbb{D}$
- D_{12}: the set of vertically composable 2-cells
- D_{20}: the set of horizontally composable 1-cells
- D_{21}: the set of horizontally composable 2-cells
- D_{22}: the set of horizontally and vertically composable 2-cells
- certain functions between these sets which are source, target, left, right, unit and compositions, satisfying well-known equations.

You can see a picture[4] of a double category with elements of D_{ij} illustrated for $0 \leq i, j \leq 2$.



Remark 3.3. The 2-cells α with $L(\alpha) = id_{ULs(\alpha)}$ and $R(\alpha) = id_{URs(\alpha)}$ are called globular 2-cells of $\mathbb{D}$.

4 Weak double categories

We first start with definition of a weak double category. The main idea is that we would like to weaken the composition of horizontal 1-cells in a coherent way. The definition below is adopted from (Shulman 2009).

Definition 4.1. A ***weak double category*** $\mathbb{D}$ consists of a pair of categories $\mathcal{D}_0$ and $\mathcal{D}_1$ and structure functors as in definition 3.1 except that laws of unit and associativity of horizontal

[4] courtesy of (Paoli 2017)

1-cells hold only up to isomorphisms. That is there are canonical globular iso 2-cells

$$\mathfrak{a} : (M \odot N) \odot P \xrightarrow{\cong} M \odot (N \odot P)$$
$$\mathfrak{l} : U_A \odot M \xrightarrow{\cong} M$$
$$\mathfrak{r} : M \odot U_B \xrightarrow{\cong} M$$

such that the standard coherence axioms similar to those for a monoidal category or bicategory hold.

REMARK *4.2.* Notice that both categories $\mathcal{D}_1$ and $\mathcal{D}_0$ in the definition are allowed to be large unlike the case of double categories.

REMARK *4.3.* In a bicategory an object is not necessarily isomorphic to itself. However, in a weak double category every object is isomorphic to itself via vertical identity morphism.

Warning!

Since most of interesting double categories we will consider are going to be weak, we will usually drop the adjective weak and by double category we really mean weak double category.

EXAMPLE *4.4.* Suppose in $\mathcal{V}$ coequalizers are preserved by its tensor product. We now form a double category of bimodules in $\mathcal{V}$. Let $\mathbb{M}\mathbf{od}(\mathcal{V})$ be the double category defined as follows. Its objects are monoids in $\mathcal{V}$ and its vertical morphisms are monoid homomorphisms. A 1-cell $M : A \nrightarrow B$ is an (A, B)-bimodule, and a 2-cell

$$\begin{tikzcd} A \arrow[r, "M", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha\Downarrow"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd}$$

is an (f, g)-*bilinear map* from M to N, that is a morphism $\alpha\colon M \to N$ in $\mathcal{V}$ which renders diagram below commutative:

$$\begin{tikzcd} A \otimes M \otimes B \arrow[r, "f\otimes\alpha\otimes g"] \arrow[d, "m"'] & C \otimes N \otimes D \arrow[d, "n"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Equivalently, α can be considered as a map of of (A, B)-bimodules $M \to {}_fN_g$ where ${}_fN_g$ is N regarded as an (A, B)-bimodule by means of action map $n \circ (f \otimes N \otimes g)$. The horizontal composition of bimodules $M\colon A \nrightarrow B$ and $N\colon B \nrightarrow C$ is given by their tensor product, so $M \odot N := M \otimes_B N$. For 2-cells

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha\Downarrow"] & B \arrow[r, "N"] \arrow[d, "g"] \arrow[dr, phantom, "\beta\Downarrow"] & E \arrow[d, "h"] \\ C \arrow[r, "P"'] & D \arrow[r, "Q"'] & F \end{tikzcd}$$

we define $\alpha \odot \beta$ to be the composite

$$M \otimes_B N \xrightarrow{\alpha \otimes_B \beta} {}_fP_g \otimes_B {}_gQ_h \twoheadrightarrow {}_fP \otimes_D Q_h \cong {}_f(P \otimes_D Q)_h\ .$$

where the middle epi is constructed as a morphism between coequalizers in below:

$$\begin{tikzcd} {}_fP_g \otimes B \otimes {}_gQ_h \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "1 \otimes g \otimes 1"'] & {}_fP_g \otimes {}_gQ_h \arrow[r, two heads] \arrow[d, "1"'] & {}_fP_g \otimes_B {}_gQ_h \arrow[d, dashed] \\ {}_fP \otimes D \otimes Q_h \arrow[r, shift left] \arrow[r, shift right] & {}_fP \otimes Q_h \arrow[r, two heads] & {}_fP \otimes_D Q_h \end{tikzcd}$$

Remark *4.5*. For a symmetric monoidal category $(\mathcal{V}, \otimes, I, \sigma)$, we define the double category $\mathbb{CM}\mathbf{od}(\mathcal{V})$ whose objects are commutative monoids, whose vertical arrows are monoid morphisms, whose horizontal 1-cells are bimodules and whose 2-cells are compatible bilinear maps of bimodules as above. $\mathbb{CM}\mathbf{od}(\mathcal{V})$ is a sub-double category of $\mathbb{M}\mathbf{od}(\mathcal{V})$. If $\mathcal{V} = \mathbf{Mod}_R$ is the monoidal category of (right) modules over a commutative ring R, then the resulting double category $\mathbb{M}\mathbf{od}(\mathcal{V})$ is comprised of internal R-algebras, R-algebra homomorphisms, and bimodules over R-algebras.

5 Framed bicategories

To every weak double category $\mathbb{D} = (L, R\colon \mathcal{D}_1 \rightrightarrows \mathcal{D}_0)$ we can associate a bicategory $\mathcal{H}(\mathbb{D})$ whose 0-cells are objects of $\mathbb{D}$, 1-cells are horizontal 1-cells of $\mathbb{D}$, and whose 2-cells are globular 2-cells in $\mathbb{D}$. $\mathcal{H}(\mathbb{D})$ is called the horizontal bicategory of $\mathbb{D}$.

Definition *5.1*. For an object D in double category $\mathbb{D}$ we define slice category $\mathcal{D}_1/D$ as the (strict) pullback of categories

$$\begin{tikzcd} \mathcal{D}_1/D \arrow[r] \arrow[d, "R^*D"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "D"] \\ \mathcal{D}_1 \arrow[r, "R"'] & \mathcal{D}_0 \end{tikzcd}$$

Objects of $\mathcal{D}_1/D$ are horizontal morphisms with codomain D and its morphisms are 2-cells α in $\mathbb{D}$ with $R(\alpha) = 1_D$. So, a morphism $\alpha\colon M \to N$ in $\mathcal{D}_1/D$ looks like

$$\begin{tikzcd} A \arrow[r, "M", "|" marking] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha\Downarrow"] & D \arrow[d, equal] \\ B \arrow[r, "N"', "|" marking] & D \end{tikzcd}$$

The following definition is taken from (Shulman 2009):

Definition 5.2. We say a double category $\mathbb{D} = (L, R\colon \mathcal{D}_1 \rightrightarrows \mathcal{D}_0)$ is a ***framed bicategory*** if $(L, R)\colon D_1 \to D_0 \times D_0$ is a fibration.

Let's spell out this definition. The fact that (L, R) is a fibration implies that

$$\begin{tikzcd} A \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd}$$

can be completed to a cartesian 2-cell (w.r.t functor (L, R))

$$\begin{tikzcd} A \arrow[r, "f^*Ng^*", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\lambda\Downarrow"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd}$$

Of course λ being cartesian implies that given any 2-cell $\beta\colon M \Rightarrow N$ with $L(\beta) = fu$ and $R(\beta) = gv$ there is a unique 2-cell γ such that

$$\begin{tikzcd} A' \arrow[r, "M", "|" marking] \arrow[d, "u"'] \arrow[dr, phantom, "\gamma\Downarrow"] & B' \arrow[d, "v"] \\ A \arrow[r, "{}_fN_g", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\lambda\Downarrow"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A' \arrow[r, "M", "|" marking] \arrow[d, "fu"'] \arrow[dr, phantom, "\beta\Downarrow"] & B' \arrow[d, "gv"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd}$$

We call ${}_fN_g = f^*Ng^*$ *restriction* of N along f and g.

Remark 5.3. Suppose $f\colon A \to C$ is a vertical arrow. There are two ways to restrict the unit U_C along f, corresponding to two cartesian 2-cell:

$$\begin{tikzcd} A \arrow[r, "{}_fC", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "{}_f\lambda\Downarrow"] & C \arrow[d, equal] \\ C \arrow[r, "U_C"', "|" marking] & C \end{tikzcd} \quad , \quad \begin{tikzcd} C \arrow[r, "C_f", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\lambda_f\Downarrow"] & A \arrow[d, "f"] \\ C \arrow[r, "U_C"', "|" marking] & C \end{tikzcd} \tag{1}$$

which are called the ***base change objects*** for restriction.[5] Based on observation above they have associated opcartesian pairs γ^f and ${}^f\gamma$ with satisfying following equations:

$$
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, equal] \arrow[dr, phantom, "\gamma^f\Downarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "{}_fC"] \arrow[d, "f"'] \arrow[dr, phantom, "{}_f\lambda\Downarrow"] & C \arrow[d, equal] \\
C \arrow[r, "U_C"'] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & A \arrow[d, "f"] \\
C \arrow[r, "U_C"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "f"'] \arrow[dr, phantom, "{}^f\gamma\Downarrow"] & A \arrow[d, equal] \\
C \arrow[r, "C_f"] \arrow[d, equal] \arrow[dr, phantom, "\lambda_f\Downarrow"] & A \arrow[d, "f"] \\
C \arrow[r, "U_C"'] & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & A \arrow[d, "f"] \\
C \arrow[r, "U_C"'] & C
\end{tikzcd}
\tag{2}
$$

Proposition *5.4*. (Shulman 2009) Four 2-cells ${}_f\lambda$, γ^f, λ_f, ${}^f\gamma$ satisfy further equations:

$$
\begin{tikzcd}
A \arrow[rr, "{}_fC"] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & C \arrow[d, equal] \\
A \arrow[r, "U_A"] \arrow[d, equal] \arrow[dr, phantom, "\gamma^f\Downarrow"] & A \arrow[r, "{}_fC"] \arrow[d, "f"] \arrow[dr, phantom, "{}_f\lambda\Downarrow"] & C \arrow[d, equal] \\
A \arrow[r, "{}_fC"'] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & C \arrow[r, "U_C"'] & C \arrow[d, equal] \\
A \arrow[rr, "{}_fC"'] & & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "{}_fC"] \arrow[d, equal] & C \arrow[d, equal] \\
A \arrow[r, "{}_fC"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "C_f"] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & A \arrow[d, equal] \\
C \arrow[r, "{}_fC"] \arrow[d, equal] \arrow[dr, phantom, "\lambda_f\Downarrow"] & A \arrow[r, "U_A"] \arrow[d, "f"] \arrow[dr, phantom, "{}^f\gamma\Downarrow"] & A \arrow[d, equal] \\
C \arrow[r, "U_C"'] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & C \arrow[r, "C_f"'] & A \arrow[d, equal] \\
C \arrow[rr, "C_f"'] & & A
\end{tikzcd}
=
\begin{tikzcd}
C \arrow[r, "C_f"] \arrow[d, equal] & A \arrow[d, equal] \\
C \arrow[r, "C_f"'] & A
\end{tikzcd}
\tag{3}
$$

and these equations plus two equations of (5.3) are necessary and sufficient to characterize the functor $(L, R)\colon D_1 \to D_0 \times D_0$ as a fibration of categories and also an opfibration of categories. In particular (L, R) is a fibration iff it is an opfibration. In either of these situations we get a double category $\mathbb{D} = (D_1 \rightrightarrows D_0)$.

Example *5.5*. Suppose $\mathbb{D}$ is a double category and suppose M is a horizontal 1-cell from A to B, and $f\colon A \to C$ is a vertical arrow. We claim that $C_f \odot M$ calculates the extension of M along f. For this, we need to prove $({}^f\gamma \odot U_M) \circ \mathfrak{l}_M^{-1}\colon M \Rightarrow C_f \odot M$ is an opcartesian 2-cell. It is enough to prove that for any horizontal 1-cell $N\colon C \to B$ and any 2-cell $\alpha\colon M \Rightarrow N$ with $L(\alpha) = f$ and $R(\alpha) = 1_B$, α factors through $({}^f\gamma \odot U_M) \circ \mathfrak{l}_M^{-1}$. Consider the following

[5] Sometimes ${}_fC$ is called a companion and C_f a cojoint for f.

identity of pasting double diagrams:

$$
\begin{tikzcd}
A \arrow[rr, "M"] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & B \arrow[d, equal] \\
A \arrow[r, "U_A"] \arrow[d, "f"'] \arrow[dr, phantom, "{}^{f}\gamma\Downarrow"] & A \arrow[r, "M"] \arrow[d, equal] \arrow[dr, phantom, "U_M\Downarrow"] & B \arrow[d, equal] \\
C \arrow[r, "C_f"] \arrow[d, equal] \arrow[dr, phantom, "\lambda_f\Downarrow"] & A \arrow[r, "M"'] \arrow[d, "f"] \arrow[dr, phantom, "\alpha\Downarrow"] & B \arrow[d, equal] \\
C \arrow[r, "U_C"'] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & C \arrow[r, "N"'] & B \arrow[d, equal] \\
C \arrow[rr, "U_C"'] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha\Downarrow"] & B \arrow[d, equal] \\
A \arrow[r, "N"'] & B
\end{tikzcd}
$$

To prove the identity above we use the facts that ${}^{f}\gamma$ and λ_f compose to U_f vertically and U_M and α compose to α vertically and U_f and α compose to α horizontally, and finally we use naturality of $\mathfrak{l}$, that is $\mathfrak{l}_N \circ \alpha \circ \mathfrak{l}_M^{-1} = \alpha$.

Recall that the data of a cloven Grothendieck fibration $p : E \to B$ includes structure of a cleavage, that is a choice ρ of cartesian lifts:

$$\rho : \prod_{a,b \in B} \prod_{\mathrm{Hom}_B(a,b)} \prod_{e \in E_b} \sum_{e' \in E_a} \mathcal{C}art_E(e', e)$$

where $\mathcal{C}art_E(e', e)$ denotes the set of cartesian morphisms from e' to e.

Proposition *5.6*. Let $(\mathbb{D}, \rho)$ be a cloven framed bicategory. It gives rise to a pseudofunctor $D_0^{\mathrm{op}} \to \mathfrak{Cat}$ taking each object D in $\mathbb{D}$ to $\mathcal{D}_1/D$ and a vertical arrow $g\colon C \to D$ to the functor $g^*\colon \mathcal{D}_1/D \to \mathcal{D}_1/C$ where $g^*N := N_g$ is specified by the cleavage, that is $N_g = \mathtt{fst}\,\rho\ ((A, C), (A, D))\ (1_A, g)\ N$. The action of functor g^* on a morphism (u, α) in $\mathcal{D}_1/D$ is defined by the unique 2-cell α_g shown in factorization below due to cartesianness of 2-cell λ_g :

$$
\begin{tikzcd}
M_g \arrow[rr] \arrow[d, mapsto] \arrow[drr, dotted, "\alpha_g"'] \arrow[drrr] & & M \arrow[dr, "\alpha"] & \\
(A', C) \arrow[dr, "{(u,1_C)}"'] \arrow[drrr] & & N_g \arrow[r, "\lambda_g"'] \arrow[d, mapsto] & N \arrow[d, mapsto] \\
& & (A, C) \arrow[r, "{(1_A,g)}"'] & (A, D)
\end{tikzcd}
$$

Notice that since composition of cartesian 2-cells is again cartesian, we get a unique globular iso 2-cell $(N_g)_f \cong N_{(g\circ f)}$ as depicted in below:

$$\begin{tikzcd}
A \arrow[r, "N_{(g\circ f)}"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & B \arrow[d, equal] \\
A \arrow[r, "(N_g)_f"] \arrow[d, equal] \arrow[dr, phantom, "\lambda_f\Downarrow"] & B \arrow[d, "f"] \\
A \arrow[r, "N_g"] \arrow[d, equal] \arrow[dr, phantom, "\lambda_g\Downarrow"] & C \arrow[d, "g"] \\
A \arrow[r, "N"', "|" marking] & D
\end{tikzcd}$$

Also, since $1_{U_A} = U_{1_A}$ is cartesian, we have a unique globular iso 2-cell $U_A \cong A_{1_A}$. This explains why we get a pseudofunctor and not a strict functor. Similarly, there is another pseudofunctor $D_0 \to \mathfrak{Cat}$ given by opfibration property of (L, R) whose action on arrow g gives left adjoint of g^*:

$$\begin{tikzcd}
\mathcal{D}_1/D \arrow[r, bend right, "g^*"'] \arrow[r, phantom, "\perp"] & \mathcal{D}_1/C \arrow[l, bend right, "g_!"']
\end{tikzcd}$$

Proposition *5.7.* Let $(\mathbb{D}, \rho)$ be a cloven framed bicategory. It gives rise to a pseudofunctor $D_0 \to \mathcal{H}(\mathbb{D})$ which is the identity on objects and is defined on morphisms (i.e. vertical arrows) by operation $f \mapsto {}_fC$.

$$\begin{tikzcd}
A \arrow[rr, "{}_{f;g}C", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & & C \arrow[d, equal] \\
A \arrow[r, "{}_fB", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "{}_f\lambda\Downarrow"] & B \arrow[r, "{}_gC", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "U_{{}_gC}\Downarrow"] & C \arrow[d, equal] \\
B \arrow[r, "U_B"] \arrow[d, "g"'] \arrow[dr, phantom, "U_g\Downarrow"] & B \arrow[r, "{}_gC"] \arrow[d, "g"] \arrow[dr, phantom, "{}_g\lambda\Downarrow"] & C \arrow[d, equal] \\
C \arrow[r, "U_C"', "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\cong"] & C \arrow[r, "U_C"', "|" marking] & C \arrow[d, equal] \\
C \arrow[rr, "U_C"', "|" marking] & & B
\end{tikzcd}$$

Similarly, the operation $f \mapsto C_f$ defines a contravariant pseudofunctor $D_0^{\mathrm{op}} \to \mathcal{H}(\mathbb{D})$.

A monoidal category can be considered as a globular double category with one object:

Definition *5.8.* Suppose $\mathbb{D}$ and $\mathbb{E}$ are weak double categories. A ***lax double functor*** $F\colon \mathbb{D} \to \mathbb{E}$ consists of the following.

- Functors $F_0 \colon \mathcal{D}_0 \to \mathcal{E}_0$ and $F_1 \colon \mathcal{D}_1 \to \mathcal{E}_1$ such that $L \circ F_1 = F_0 \circ L$ and $R \circ F_1 = F_0 \circ R$.
- Natural transformations $F_\odot \colon F_1 M \odot F_1 N \to F_1(M \odot N)$ and $F_U \colon U_{F_0 A} \to F_1(U_A)$, whose components are globular, and which satisfy the usual coherence axioms for a lax monoidal functor or 2-functor.

Dually, we have the definition of an **oplax double functor**, for which $F_\odot$ and F_U go in the opposite direction. A **strong double functor** is a lax double functor for which $F_\odot$ and F_U are (globular) isomorphisms. If just F_U is an isomorphism, we say that F is **normal**.

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