Comma objects= cotensor with walking arrow + pullbacks with generic comma object

April 13, 2016

A 2-category $\mathcal{K}$ is a *representable 2-category* if $\mathcal{K}$ has all strict pullbacks as well as cotensors with the (free) walking arrow category[1] $\mathbb{2}$. For a 0-cell B of $\mathcal{K}$, cotensor of B with $\mathbb{2}$ is an object $B \downarrow B$ of $\mathcal{K}$ together with 1-cells $d_0, d_1 \colon B \downarrow B \rightrightarrows B$ and a (generic) 2-cell ϕ between them which is universal in the sense that pasting of ϕ with 1-cells induces an isomorphism of categories $\mathcal{K}(X, B \downarrow B) \cong \mathfrak{Cat}(\mathbb{2}, \mathcal{K}(X, B))$ naturally for 0-cell X of $\mathcal{K}$. We observe that any representable 2-category has all comma objects; suppose $A \xrightarrow{f} B \xleftarrow{g} C$ is an opspan in $\mathcal{K}$. Then the comma object $f \downarrow g$ can be constructed as the following pasting of ϕ with (canonical) pullbacks in below:

$$
\begin{array}{ccccc}
f \downarrow g & \xrightarrow{\ \overline{f}\ } & g^*(B \downarrow B) & \xrightarrow{\ g^* d_1\ } & C \\
\Big\downarrow{\scriptstyle \overline{g}} & \lrcorner & \Big\downarrow{\scriptstyle d_1^* g} & \lrcorner & \Big\downarrow{\scriptstyle g} \\
f^*(B \downarrow B) & \xrightarrow{\ d_0^* f\ } & B \downarrow B & \xrightarrow{\ d_1\ } & B \\
\Big\downarrow{\scriptstyle f^* d_0} & \lrcorner & \Big\downarrow{\scriptstyle d_0} & \phi \Uparrow & \Big\downarrow{\scriptstyle 1} \\
A & \xrightarrow[\ f\ ]{} & B & \xrightarrow[\ 1\ ]{} & B
\end{array}
$$

The proof relies on the fact that the universal property of $\langle f \downarrow g, f^* d_0 \circ \overline{g}, g^* d_1 \circ \overline{f}, \phi \centerdot (d_0^* f \centerdot \overline{g}) \rangle$ can be expressed as combinations of universal property of pullbacks and universal property of $\langle B \downarrow B, d_0, d_1, \phi \rangle$.

[1] It has only two objects and one non-trivial morphism.

Definition 0.1. A **finitely complete 2-category** is a 2-category that admits finite conical limits[2] and cotensors with the walking arrow category $\mathbb{2}$.

The above observation shows that

Proposition 0.2. A finitely complete 2-category $\mathcal{K}$ has all comma objects.

[2]i.e. weighted limits with set-valued weight functors. They are ordinary limit as opposed to a more general weighted limit.