Kripke-Joyal Forcing for Homotopy Type Theory

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Orton-Pitts Cubical Models of HoTT

Starting from

- a topos $\mathscr{E}$,
- a universe $\Phi \rightarrowtail \Omega$ in $\mathscr{E}$ of "cofibrant" propositions satisfying certain axioms (including dominance),
- a bipointed tiny object $\mathbb{I}$ in $\mathscr{E}$, and
- a universe $\mathsf{U}_\bullet \to \mathsf{U}$ in $\mathscr{E}$ of small families closed under Σ, Π and ...

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In this model,

- the interval is modelled by the object $\mathbb{I}$, and types are modelled by the "fibrant" objects of $\mathscr{E}$, and
- the path type of a type A is modelled by $A^{\mathbb{I}}$.

Quillen Model Structure from Orton-Pitts topos

From an Orton-Pitts topos (i.e. a topos $\mathscr{E}$ equipped with $(\Phi, \mathbb{I}, \mathrm{U}_\bullet)$ as above), (Awodey, 2018) constructs a QMS on $\mathscr{E}$ and shows this QMS is right proper and has descent.

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- Kripke-Joyal forcing semantics for HoTT is a useful machinery in dealing with these complexities.
- It relates the type-theoretic developments (Cohen et al., 2018), (Orton and Pitts, 2018), etc. with the diagrammatic developments (Gambino and Sattler, 2017), (Sattler, 2017), (Awodey, 2018), etc. found in the models of HoTT literature.

The universe Φ of cofibrant propositions

As in (Orton and Pitts, 2018), we consider a modality

$$\mathsf{cof} : \Omega \to \Omega$$

satisfying:

$$\mathsf{cof} \circ \mathsf{true} = \mathsf{true}$$

$$\mathsf{cof} \circ \mathsf{false} = \mathsf{true}$$

$$\forall (\varphi, \psi : \Omega).\ \mathsf{cof}\, \varphi \Rightarrow (\varphi \Rightarrow \mathsf{cof}\, \psi) \Rightarrow \mathsf{cof}(\varphi \wedge \psi)$$

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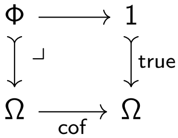
Φ as the comprehension subtype in the internal language:

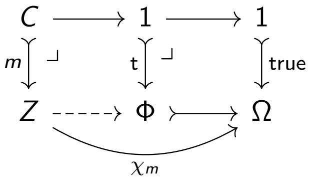
$$\Phi \triangleq \{\varphi \in \Omega \mid \mathsf{cof}\, \varphi\}$$



Cofibrations

Definition

A monomorphism $m\colon C \rightarrowtail Z$ is a **cofibration** if its classifying map $\chi_m\colon Z \to \Omega$ factors through $m_{\mathrm{cof}}\colon \Phi \rightarrowtail \Omega$.



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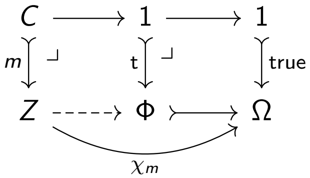

All cofibrations are the pullbacks of the generic cofibration $\mathrm{t} \colon 1 \rightarrowtail \Phi$.

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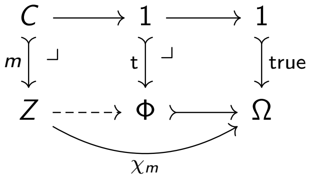

Proposition

$m\colon C \rightarrowtail Z$ *is a cofibration* $\Leftrightarrow \mathscr{E} \Vdash \forall z : Z,\ \mathrm{cof}(\exists c : C,\ m(c) = z)$.

Forcing for partial elements (I)

We define $(-)^{+} \colon \mathscr{E} \to \mathscr{E}$ to be the polynomial endofunctor associated to the map $\mathsf{t} \colon 1 \to \Phi$, namely the composite

$$\mathscr{E} \xrightarrow{\mathsf{t}_*} \mathscr{E}_{/\Phi} \xrightarrow{\Phi_!} \mathscr{E}\,.$$

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If A is classified by type α, then A^{+} is classified by

$$\alpha^{+} := \textstyle\sum_{\varphi\,:\,\Phi} \{\varphi\} \to \alpha\,,$$

We call α^{+} the **type of cofibrant partial elements** of a type α

Forcing for cofibrant partial elements (II)

Proposition (Forcing for partial elements)

Let $\alpha \colon X \to \mathsf{U}$ *and* $x \colon \mathrm{y}(c) \to X$. *Then the following conditions are equivalent.*

1. $c \Vdash (\varphi, u) : \alpha^{+}(x)$
2. $c \Vdash \varphi(x) : \Phi$ *and for every* $f \colon d \to c$, *if* $d \Vdash p : \{\varphi\}(xf)$ *then* $d \Vdash \mathrm{app}(u_f, p) : \alpha(xf)$, *and furthermore the following uniformity condition holds:*

$$\mathrm{app}(u_f, p)g = \mathrm{app}(u_{fg}, p)$$

for any $g \colon e \to d$ *in* $\mathcal{C}$.

Uniform trivial fibrations

- The subcategory of $\mathscr{E}^{\to}$ consisting of cofibrations and cartesian squares between them determines an algebraic weak factorisation system $(\mathsf{Cof}, \mathsf{TrivFib})$ on $\mathscr{E}$ (Gambino and Sattler, 2017).

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- These structured maps can also be described in the internal type theory of $\mathscr{E}$ (Orton and Pitts, 2018).

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- The maps in the right class $(\mathrm{TrivFib})$ are the uniform trivial fibrations which appear in the semantics of Homotopy Type Theory.
- These structured maps can also be described in the internal type theory of $\mathscr{E}$ (Orton and Pitts, 2018).
- We shall see how the uniformity condition arises naturally from Kripke-Joyal forcing.

Uniform trivial fibrations

A **uniform trivial fibration structure** on a small map $p\colon A \to X$ assigns

$$\begin{array}{c} A \\ \Big\downarrow{\scriptstyle p} \\ X \end{array}$$

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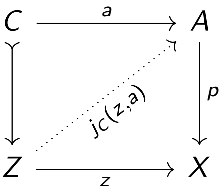
Uniform trivial fibrations

A **uniform trivial fibration structure** on a small map $p\colon A \to X$ assigns to every cofibration $C \rightarrowtail Z$ and to every commutative square

$$\begin{array}{ccc} C & \xrightarrow{\;a\;} & A \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[\;z\;]{} & X \end{array}$$

Uniform trivial fibrations

A **uniform trivial fibration structure** on a small map $p\colon A \to X$ assigns to every cofibration $C \rightarrowtail Z$ and to every commutative square a diagonal filler $j_C(z,a)\colon Z \to A$, subject to the following **uniformity** condition:



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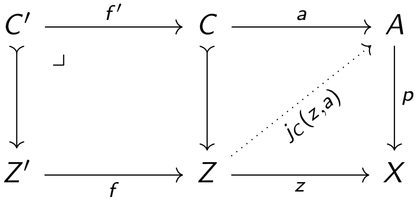

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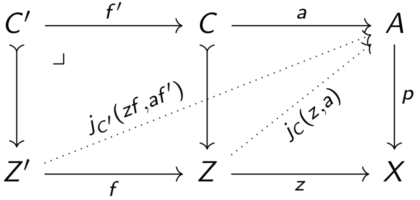
$$j_{C'}(zf, af') = j_C(z,a) \circ f\ .$$



The type of uniform trivial fibration structures

For any type $\alpha\colon X \to \mathsf{U}$ define

$$\mathsf{TFib}(\alpha) := \prod_{\varphi:\Phi} \prod_{u:\{\varphi\}\to\alpha} \sum_{a:\alpha} (u =^{\varphi} a)\,,$$

where the type $(u =^{\varphi} a)$ is defined

$$(u =^{\varphi} a) := \prod_{p:\{\varphi\}} \mathsf{app}(u,p) =_{\alpha} a\,.$$

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See Extension Types of (Riehl and Shulman, 2017) and (Orton and Pitts, 2018).

The type of uniform trivial fibration structures

Theorem

The display map $p_\alpha : X.\alpha \to X$ *is a* **uniform** *trivial fibration* $\Leftrightarrow$ *there is a term* $X \vdash t : \mathsf{TFib}(\alpha)$*.*

Since uniform trivial fibrations are expressed in terms of diagrams, and since the type TFib(α) has a bunch of Π and Σ, we need a diagrammatic unfolding of KJ-forcing of Π and Σ types involved in TFib(α).

Diagrams for Kripke-Joyal forcing: Definition

Definition

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X

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$$X \xrightarrow[\alpha]{} U$$

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- For a context X,
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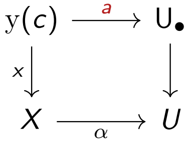
  we say that c **forces** $a : \alpha(x)$, written as $c \Vdash a : \alpha(x)$,
  if the square commutes.



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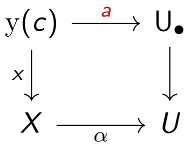
- For $a, b\colon \mathrm{y}(c) \to \mathrm{U}_\bullet$ such that $c \Vdash a\colon \alpha(x)$ and $c \Vdash b\colon \alpha(x)$, we say that c **forces** $a = b\colon \alpha(x)$, written $c \Vdash a = b\colon \alpha(x)$, if a and b are equal maps in $\mathscr{E}$.



Diagrams for Kripke-Joyal forcing: Σ types

Proposition

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Diagrams for Kripke-Joyal forcing: Σ types

Proposition

Given a context X, a type α in context X,

$$X.\alpha \xrightarrow{p_\alpha} X$$

Diagrams for Kripke-Joyal forcing: Σ types

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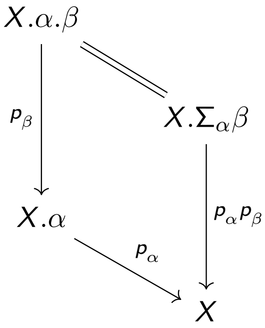
Given a context X, a type α in context X, a type β in context $X \cdot \alpha$,



Diagrams for Kripke-Joyal forcing: Σ types

Proposition

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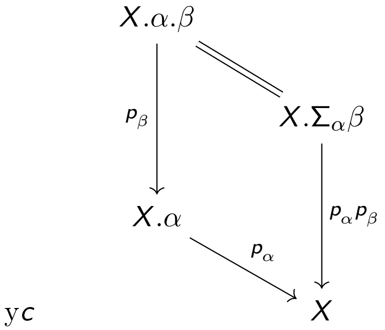

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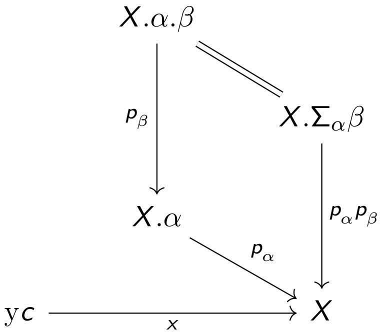


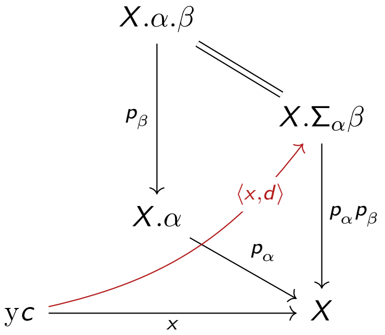
Diagrams for Kripke-Joyal forcing: Σ types

Proposition

Given a context X, a type α in context X, a type β in context $X \cdot \alpha$, an object c of $\mathcal{C}$, and a morphism $x\colon \mathrm{y}c \to X$,

$$c \Vdash d : (\Sigma_\alpha \beta)(x)$$

iff



Diagrams for Kripke-Joyal forcing: Σ types

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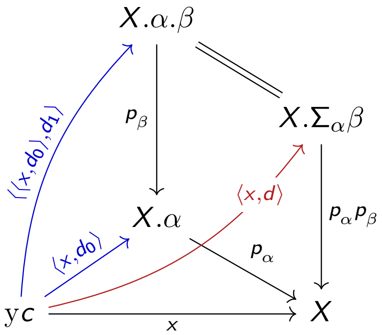
iff

$$d = (d_0, d_1)$$
$$c \Vdash d_0 : \alpha(x)$$
$$c \Vdash d_1 : \beta(\langle x, d_0 \rangle) \ .$$



Diagrams for Kripke-Joyal forcing: Π types

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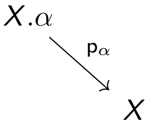

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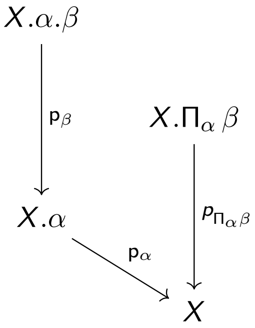

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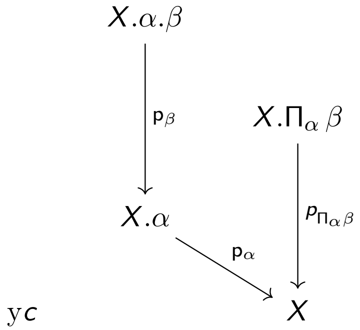

Diagrams for Kripke-Joyal forcing: Π types

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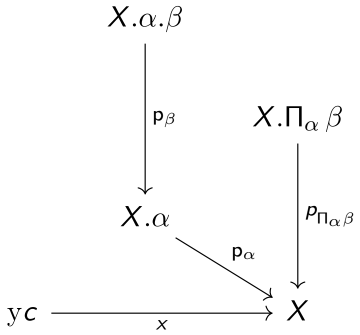

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$$c \Vdash b : (\Pi_\alpha \beta)(x)$$

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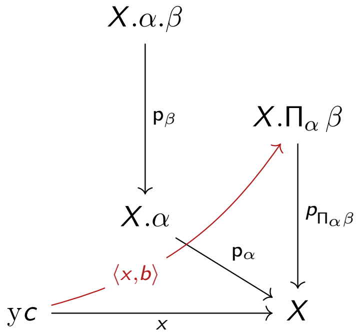

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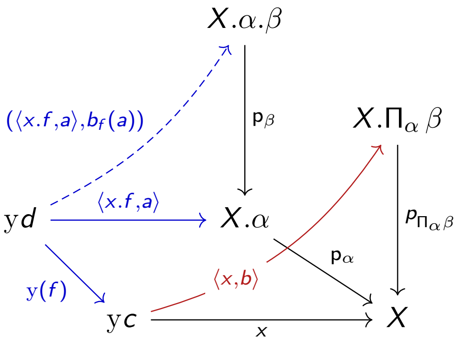
iff there is a function b such that for every morphism $f : d \to c$ in $\mathcal{C}$, if

$$d \Vdash a : \alpha(x.f)$$

then

$$d \Vdash b_f(a) : \beta(\langle x.f, a\rangle)$$

and for every $g\colon d' \to d$, $b_f(a).g = b_{f\circ g}(a.g)$.



Uniform trivial fibration structure via Kripke-Joyal forcing (I)

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Lemma

Suppose $\alpha : X \to U$ and $x : \mathrm{y}c \to X$. Given

$$c \Vdash a : \alpha(x)$$

$$c \Vdash \varphi : \Phi$$

$$c \Vdash u : \big(\{\varphi\} \to \alpha\big)(x)$$

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we have

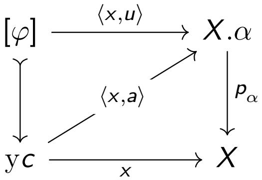
$$c \Vdash e : (u =_{\varphi} a)(x) \quad \Leftrightarrow \quad \begin{array}{ccc} [\varphi] & \xrightarrow{\langle x,u\rangle} & X.\alpha \\ \downarrow & \nearrow^{\langle x,a\rangle} & \downarrow p_\alpha \\ \mathrm{y}c & \xrightarrow[x]{} & X \end{array} \quad \textit{commute.}$$

Proof of Lemma.

- $c \Vdash a : \alpha(x)$

  $\Leftrightarrow$

  the lower triangle commutes.



Proof of Lemma.

- $c \Vdash a : \alpha(x)$
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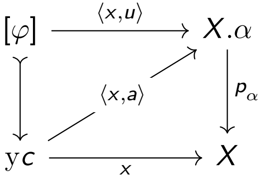

Proof of Lemma.

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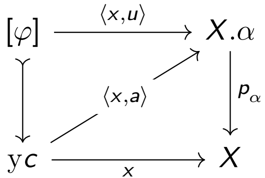
  outer square commutes.
- $c \Vdash e : (u =^{\varphi} a)(x) \Leftrightarrow$
  $c \Vdash e : \prod_{p:[\varphi]} (\mathrm{app}(u,p) =_{\alpha} a)(x)$
  $\Leftrightarrow$
  for all $f : d \to c$, if
  $d \Vdash p : \{\varphi\}(x.f)$ then
  $d \Vdash e_f(p) : (\mathrm{app}(u,p) =_{\alpha} a)(x.f)$
  $\Leftrightarrow$
  the top triangle commutes (by the
  Yoneda lemma). QED.



Proof of the uniform trivial fibration forcing theorem

Recall that

$$\begin{aligned}\mathsf{TFib}(\alpha) &= \prod_{\varphi:\Phi} \prod_{u:[\varphi]\to\alpha} \sum_{a:\alpha} (u =_\varphi a) \\ &= \prod_{(\varphi,u)\,:\,\alpha^+} \sum_{a:\alpha} u =_\varphi a\end{aligned}$$

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Hence,

$$c \Vdash t_x : \prod_{(\varphi,u)\,:\,\alpha^+} \sum_{a:\alpha} (u =_\varphi a)(x)\ .$$

Proof of Theorem (cont'd)

By Kripke–Joyal semantics of $\prod$ and $\sum$, we have that for every $f : d \to c$ in $\mathcal{C}$, if

$$d \Vdash (\varphi, u) : \alpha^{+}(x.f) \tag{1}$$

then

$$d \Vdash t_{x.f}(\varphi, u)^{0} : \alpha(x.f) \tag{2}$$

and

$$d \Vdash t_{x.f}(\varphi, u)^{1} : \left(u =^{\varphi} t_{x.f}(\varphi, u)^{0}\right)(x.f) \tag{3}$$

and, for any $g : d' \to d$,

$$t_{x.f}(\varphi, u).g = t_{(x.fg)}(\varphi.g, u.g)\,. \tag{4}$$

Proof of Theorem (cont'd)

Unfolding the condition (1)

$$d \Vdash (\varphi, u) : \alpha^{+}(x.f)$$

$$\begin{array}{ccc} [\varphi.f] & \xrightarrow{\langle x.f, u_f \rangle} & X.\alpha \\ \big\downarrow & & \big\downarrow p_\alpha \\ \mathrm{y}d & \xrightarrow[x.f]{} & X \end{array}$$

Proof of Theorem (cont'd)

Unfolding the condition (1)

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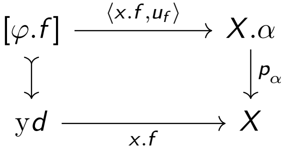
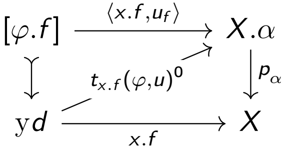


Lemma applied to (2) and (3)

$$d \Vdash t_{x.f}(\varphi, u)^0 : \alpha(x.f)$$

$$d \Vdash t_{x.f}(\varphi, u)^1 : \big(u =^{\varphi} t_{x.f}(\varphi, u)^0\big)(x.f)$$

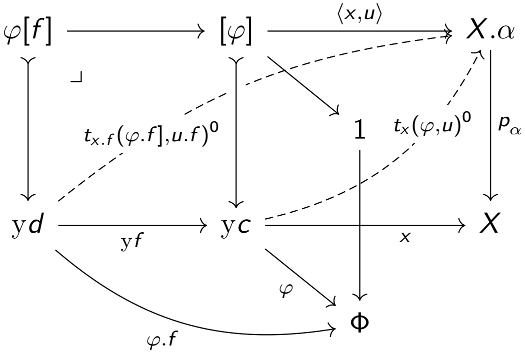


Proof of Theorem (cont'd)

Thus forcing TFib(α) produces diagonal fillers

$$j_\varphi(x, u) \triangleq t_{x.f}(\varphi, u)^0$$

for each lifting problem as in the right hand square below:



Proof of Theorem (cont'd)

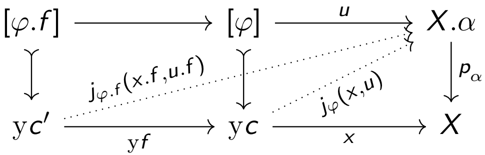
And equation (4)

$$t_{x.f}(\varphi, u).g = t_{(x.fg)}(\varphi.g, u.g)$$

guarantees the uniformity of the lifts j.

Proof of Theorem (cont'd) – the converse

If $p_\alpha : X.\alpha \to X$ is a uniform trivial fibration then, in particular, we have diagonal filler for *basic* cofibrations:



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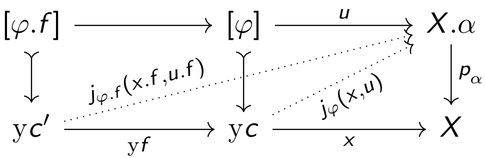
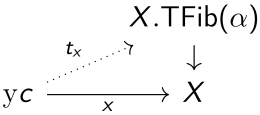


By the lemma, this corresponds to an element $t_x : \mathrm{y}c \to \mathrm{TFib}(\alpha)$ over $x : \mathrm{y}c \to X$,



The uniformity condition says exactly that for all $f : c' \to c$, the elements t_x cohere, $t_{(x.f)} = t_x.f$.

By Yoneda for the slice category $\mathscr{E}/X$, there is a section of $X.\mathrm{TFib}(\alpha) \to X$, i.e. there is a term $X \vdash t : \mathrm{TFib}(\alpha)$. QED.

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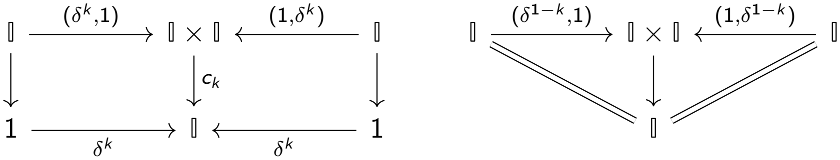
The interval

An **interval with connections** is a presheaf $\mathbb{I}$ in $\mathscr{E}$ equipped with endpoints, i.e. maps $\delta^k \colon 1 \to \mathbb{I}$, for $k \in \{0,1\}$, and connections, i.e. maps $c_k \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ for $k \in \{0,1\}$, satisfying the following axioms.

1. $\delta^0 \neq \delta^1$
2. $\delta^k \colon 1 \to \mathbb{I}$ is a cofibration, for $k \in \{0,1\}$.
3. The diagrams



commute, for $k \in \{0,1\}$.

Naive trivial cofibrations

A **naive trivial cofibrations** is a Leibniz tensors of the form

$$c \otimes \delta_k : Z +_C (Z \times \mathbb{I}) \rightarrowtail Z \times \mathbb{I}\,,$$

for an arbitrary cofibration $c\colon C \rightarrowtail Z$ and an endpoint $\delta_k\colon 1 \rightarrowtail \mathbb{I}$, for $k \in \{0, 1\}$.

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for an arbitrary cofibration $c \colon C \rightarrowtail Z$ and an endpoint $\delta_k \colon 1 \rightarrowtail \mathbb{I}$, for $k \in \{0, 1\}$. This construction is stable under pullback, in the sense that for any map $t \colon Z' \to Z$, one has a pullback square

$$\begin{array}{ccc} Z' +_{C'} (C' \times \mathbb{I}) & \longrightarrow & Z +_C (C \times \mathbb{I}) \\ {\scriptstyle (t^* c)\otimes\delta_k} \big\downarrow & & \big\downarrow {\scriptstyle c\otimes\delta_k} \\ Z' \times \mathbb{I} & \xrightarrow[h\times\mathbb{I}]{} & Z \times \mathbb{I} \end{array}$$

Uniform fibration structure

A **uniform fibration structure** on a small map $p\colon A \to X$ consists of a function j that assigns a dotted filler $j(i, u, v)\colon Z \times \mathbb{I} \to A$ to every diagram of solid arrows

$$\begin{array}{ccc} Z +_C (C \times \mathbb{I}) & \xrightarrow{u} & A \\ {\scriptstyle m \otimes \delta_k} \downarrow & & \downarrow \\ Z \times \mathbb{I} & \xrightarrow[v]{} & X \end{array}$$

where c is a cofibration and $k \in \{0, 1\}$, subject to the following *uniformity condition*: for any map $t\colon Z' \to Z$ and induced pullback square on the left,

$$\begin{array}{ccccc} Z +'_C (C' \times \mathbb{I}) & \longrightarrow & Z +_C (C \times \mathbb{I}) & \xrightarrow{u} & A \\ {\scriptstyle c' \otimes \delta} \downarrow & & \downarrow & & \downarrow \\ Z' \times \mathbb{I} & \xrightarrow[h \times \mathbb{I}]{} & Z \times \mathbb{I} & \xrightarrow[v]{} & X, \end{array} \tag{1}$$

we have that $j(i, uh', vh) = j(i, u, v) \circ (h \times \mathbb{I})$.

The type of fillings

Let $\alpha : \mathbb{I} \to \mathrm{U}$.

Recall the **type of 0-directed filling structure**

$$\mathrm{Fill}_0(\alpha) = \prod_{\varphi : \Phi} \prod_{u : \{\varphi\} \to \prod_{i : \mathbb{I}} \alpha(i)} \prod_{a : \alpha_0} (u_0 =^{\varphi} a) \to \sum_{s : \prod_{i : \mathbb{I}} \alpha(i)} (s_0 =_{\alpha_0} a) \times (u =^{\varphi} s)\,,$$

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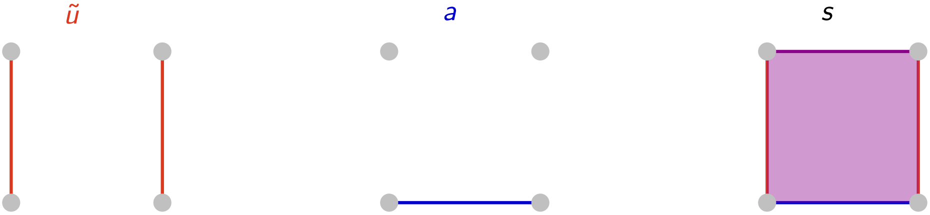
Example:

For $i : \mathbb{I} \vdash \varphi = (i = 0 \vee i = 1)$



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For any type $X : \mathrm{U}$ and family of types $\alpha : X \to \mathrm{U}$, we then define the **type of fibration structures**

$$\mathrm{Fib}(\alpha) := \prod_{x : \mathbb{I} \to X} \mathrm{Fill}_0(\alpha(x)) \times \mathrm{Fill}_1(\alpha(x)) \,.$$

Forcing for type Fib

Theorem

Let $\alpha \colon X \to \mathsf{U}$. *Then the following conditions are equivalent.*

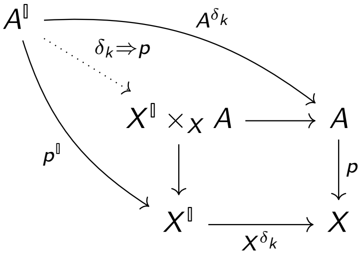
1. *The map* $p_\alpha \colon X.\alpha \to X$ *is a uniform fibration.*
2. *There is a term* $t : \mathsf{Fib}(\alpha)$.

(TrivCof, Fib) from (Cof, TrivFib)

On the arrow category $\mathscr{E}^{\to}$,

$$(-) \otimes \delta_k \dashv \delta_k \Rightarrow (-)$$

the pullback-hom, taking $p \colon A \to X$ to the map $\delta_k \Rightarrow p$ indicated in the following diagram.



By adjointness, these operations can be seen to satisfy

$$(m \otimes \delta_k) \pitchfork p \quad \text{if and only if} \quad m \pitchfork (\delta_k \Rightarrow p)$$

naturally in m and p.

Universal uniform fibration (I)

Unfortunately the type

$$\mathsf{Fib}(\alpha) := \prod_{x\,:\,\mathbb{I}\to X} \mathsf{Fill}_0(\alpha(x)) \times \mathsf{Fill}_1(\alpha(x))\,.$$

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Using the amazing right adjoint, we define the universal fibration $\mathsf{Fib}^* \to U$ as the pullback of $\mathsf{U}_\bullet \to U$ along $(\mathsf{Fill})_{\mathbb{I}} \circ \eta$.

$$\begin{array}{ccccccc}
\mathsf{Fib}^*(\alpha) & \longrightarrow & \mathsf{Fib}^* & & \longrightarrow & & (\mathsf{U}_\bullet)_{\mathbb{I}} \\
\downarrow & & \downarrow & \lrcorner & & & \downarrow \\
X & \xrightarrow[\alpha]{} & U & \xrightarrow[\eta]{} & (U^{\mathbb{I}})_{\mathbb{I}} & \xrightarrow[(\mathsf{Fill})_{\mathbb{I}}]{} & (U)_{\mathbb{I}}
\end{array}$$

Universal uniform fibration (II)

Theorem

Let $\alpha\colon X \to \mathsf{U}$.

1. *There is a bijection between points* $1 \to \mathsf{Fib}(\alpha)$ *and sections of* $\mathsf{Fib}^*(\alpha)$ *over* X, *which is natural in* X.
2. *The object* $\mathsf{Fib}^*(\alpha)$ *is stable under pullback along any map* $\gamma\colon \Delta \to X$.

$$
\begin{array}{ccc}
\Delta.\mathsf{Fib}^*(\alpha(\gamma)) & \longrightarrow & X.\mathsf{Fib}^*(\alpha) \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow[\gamma]{} & X \\
\uparrow & & \uparrow \\
\Delta.\alpha(\gamma) & \longrightarrow & X.\alpha
\end{array}
$$

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The end!

Thanks for your attention!