Kripke-Joyal Semantics for Dependent Type Theory[1]

MURI seminars

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Outline

1. The internal dependent type theory of presheaf toposes
2. Dependent Kripke–Joyal semantics
3. Cofibrations and uniform trivial fibrations
4. Trivial cofibrations and uniform fibrations

The Main Motivation

- We want a formal, sound, and *mechanical* process to relate internal developments (Cohen et al., 2018), (Orton and Pitts, 2018), (Licata et al., 2018), etc. with the diagrammatic developments (Gambino and Sattler, 2017), (Sattler, 2017), (Awodey, 2018), etc. found in the models of HoTT literature.
- We achieve this by developing a Kripke–Joyal semantics for dependent type theories including HoTT.

Features of dependent Kripke–Joyal semantics

- By restricting to propositions, we recover the Kripke–Joyal semantics for the internal Intuitionistic Higher Order Logic (IHOL) of toposes.(Boileau and Joyal, 1981), (Mac Lane and Moerdijk, 1994)
- Provability versus proof relevance: the classical Kripke–Joyal semantics is only concerned with the provability of a proposition; it is proof irrelevant. Kripke–Joyal semantics for dependent type theory includes the terms (proofs) in the forcing statements.
- Unbounded quantification via universes vs bounded quantification in IHOL of toposes.
- Equality of terms "up to homotopy" instead of extensional equality in IHOL of toposes.

1. The internal dependent type theory of presheaf toposes
   - Fixing the setup
   - CwF structure on $\mathscr{E} = [\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}]$
   - Basic types and type formers of the internal dependent type theory
   - The impredicative universe of propositions

Fixing the setup (I)

- We fix a small category $\mathcal{C}$.
- We define the topos of presheaves $\mathscr{E} = [\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathrm{et}]$.
- We write $\mathrm{y}\colon \mathcal{C} \to \mathscr{E}$ for the Yoneda embedding.
- We assume a Grothendieck universe of κ-small sets in the ambient set theory.
- Using this Grothendieck universe we define **small maps** in $\mathscr{E}$.

Fixing the setup

- The category $\mathscr{E}$ admits a classifier for small maps, given by the Hofmann-Streicher universe $U \in \mathscr{E}$.
$$U(c) := \mathrm{Obj}\left[(\mathcal{C}_{/c})^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\kappa}\right]$$
- We get a small-map classifier
$$\pi \colon E \to U$$
where
$$E(c) := \mathrm{Obj}\left[(\mathcal{C}_{/c})^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\kappa}^{\bullet}\right].$$
- π is small.
- For every small map $p \colon A \to \Gamma$, there exists a canonical α fitting in the pullback diagram
$$\begin{tikzcd} A \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\ \Gamma \arrow[r, "\alpha"'] & U . \end{tikzcd}$$

CwF structure on $\mathscr{E}$ (I)

Following (Awodey, 2018), we get a CwF structure on $\mathscr{E}$ from the universe $\pi\colon E \to U$:

The **contexts** are the objects of $\mathscr{E}$.

The **substitutions** are morphisms in $\mathscr{E}$ from arbitrary natural transformations.

A **type** α in a context Γ is a map $\alpha\colon \Gamma \to U$.

A **term** $a\colon \alpha$ in context Γ is a map $a\colon \Gamma \to E$ with $\pi \circ a = \alpha$.

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ \Gamma \arrow[ur, dotted, "a"] \arrow[r, "\alpha"'] & U \end{tikzcd}$$

The **context extension** is given by the pullback along π.

$$\begin{tikzcd} \Gamma.\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\ \Gamma \arrow[r, "\alpha"'] & U \end{tikzcd}$$

CwF structure on $\mathscr{E}$ (II)

$$\frac{\alpha \in \mathsf{Type}(\Gamma) \qquad x \colon \Delta \to \Gamma}{\alpha(x) \triangleq \alpha \circ x \in U(\Delta)}\ \text{(substitution-types)}$$

$$\frac{a \in \mathsf{Term}(\Gamma, \alpha) \qquad x \colon \Delta \to \Gamma}{a(x) \triangleq a \circ x \in \mathsf{Term}(\Delta, \alpha(x))}\ \text{(substitution-terms)}$$

$$\begin{tikzcd}
 & & E \arrow[d, "\pi"] \\
\Delta \arrow[r, "x"'] & \Gamma \arrow[ur, dotted, "a"] \arrow[r, "\alpha"'] & U
\end{tikzcd}$$

When $\Delta = \mathrm{y}c$:
by the Yoneda lemma, $x \in \Gamma(c)$, and $\alpha(x) \in U(c)$.

For a morphism $f \colon d \to c$ in $\mathcal{C}$, we write $x.f$ for the composite $x \circ \mathrm{y}f$.
Also, for $a \in \mathsf{Term}(\mathrm{y}c, \alpha(x))$ and we have $a(x.f) = a(x).f \in \mathsf{Term}(d, \alpha(x.f))$.

Basic types and type formers of the internal dependent type theory

$\mathcal{T}(\mathscr{E})$ = the internal dependent type theory of presheaf toposes.

$\mathcal{T}(\mathscr{E})$ has

- the basic types 0, 1, $\mathbf{2}$, $\mathbb{N}$ (as well as many other inductively defined types).
- also the following forms of type:

$$\alpha \times \beta\,, \quad \alpha + \beta\,, \quad \alpha \to \beta$$
$$a =_\alpha b\,, \quad \sum_{x\,:\,\alpha} \beta(x)\,, \quad \prod_{x\,:\,\alpha} \beta(x)$$

- The type of extensional equality $\_ =_\alpha \_$ is given by the diagonal map $\alpha \rightarrowtail \alpha \times \alpha$.
- These types satisfy the usual induction and computation rules, e.g. in HoTT-Book.
- There is an evident tautological interpretation of $\mathcal{T}(\mathscr{E})$ into $\mathscr{E}$, using the CwF structure of $\mathscr{E}$.

The impredicative universe Ω of propositions

$\mathcal{T}(\mathscr{E})$ additionally has the impredicative universe Ω of propositions given by

$$\Omega(c) := \mathrm{Obj}\big[(\mathcal{C}_{/c})^{\mathrm{op}}, \mathbb{2}\big]$$

- The inclusion $\mathbb{2} \hookrightarrow \mathcal{S}\mathrm{et}_\kappa$ induces a monomorphism $\{-\} \colon \Omega \rightarrowtail U$ in $\mathscr{E}$.
- The subobject classifier $\mathsf{true} \colon 1 \to \Omega$ fits into the pullback square on the right.
- $\Gamma.\{\varphi\} = \{x \colon \Gamma \mid \varphi(x)\}$ as subobjects of Γ.

$$\begin{tikzcd}
\{x : \Gamma \mid \varphi(x)\} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[r] \arrow[d, tail, "\mathsf{true}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\
\Gamma \arrow[r, "\varphi"'] & \Omega \arrow[r, "\{-\}"'] & U
\end{tikzcd}$$

2. Dependent Kripke–Joyal semantics
   - Definition
   - Dependent Kripke–Joyal semantics generalizes Kripke–Joyal semantics
   - Forcing for dependent sum types
   - Forcing for dependent product types
   - Propositional forcing

Dependent Kripke–Joyal semantics (I)

Definition (Dependent Kripke–Joyal semantics)

- For a context Γ,
  a type α in context Γ,
  an object c of $\mathcal{C}$,
  and a morphism $x \colon \mathrm{y}c \to \Gamma$,
  we say that c **forces** $a : \alpha(x)$, written as $c \Vdash a : \alpha(x)$,
  if the square commutes.

- For $a, b \colon \mathrm{y}(c) \to E$ such that $c \Vdash a \colon \alpha(x)$ and
  $c \Vdash b \colon \alpha(x)$, we say that c **forces** $a = b \colon \alpha(x)$, written
  $c \Vdash a = b \colon \alpha(x)$, if a and b are equal maps in $\mathscr{E}$.

$$\begin{tikzcd} \mathrm{y}(c) \arrow[r, "a"] \arrow[d, "x"'] & E \arrow[d] \\ \Gamma \arrow[r, "\alpha"'] & U \end{tikzcd}$$

Dependent Kripke–Joyal semantics (II)

Proposition

$\Gamma \vdash a : \alpha \;\Leftrightarrow\;$ *There is a family* $(a_x \mid c \in \mathcal{C},\ x : \mathrm{y}c \to \Gamma)$ *satisfying*

$$c \Vdash a_x : \alpha(x)$$

and for every morphism $f : d \to c$ *of* $\mathcal{C}$*,*

$$a_x.f = a_{x.f}$$

Dependent Kripke–Joyal semantics generalizes Kripke–Joyal semantics

$$c \Vdash a\colon \alpha(x)$$

$$\begin{tikzcd}
 & \Gamma.\alpha \arrow[r] \arrow[d] & E \arrow[d] \\
\mathrm{y}(c) \arrow[r, "x"'] \arrow[ur, dotted, bend left, "{\langle x, a\rangle}"] & \Gamma \arrow[r, "\alpha"'] & U
\end{tikzcd}$$

$$c \Vdash \varphi(x)$$

$$\begin{tikzcd}
 & \{\Gamma \mid \varphi\} \arrow[r] \arrow[d] & 1 \arrow[d] \\
\mathrm{y}(c) \arrow[r, "x"'] \arrow[ur, dotted, bend left] & \Gamma \arrow[r, "\varphi"'] & \Omega
\end{tikzcd}$$

Remark

In the forcing conditions for types we need to carry around the map $\langle x, a\rangle$, as it is not unique in general.

Theorem

Let $\varphi\colon \Gamma \to \Omega$ and $x\colon \mathrm{y}(c) \to \Gamma$. Then the following are equivalent:

1. *$c \Vdash \varphi(x)$ in the sense of the standard Kripke-Joyal semantics,*
2. *there exists a (necessarily unique) $s\colon \mathrm{y}(c) \to E$ such that $c \Vdash s\colon \{\varphi\}(x)$.*

Forcing dependent sum types

Proposition

Given a context Γ*, a type* α *in context* Γ*, a type* β *in context* $\Gamma.\alpha$*, an object* c *of* $\mathcal{C}$*, and a morphism* $x\colon \mathrm{y}c \to \Gamma$*,*

$$c \Vdash d : (\Sigma_\alpha \beta)(x)$$

iff

$$d = (d_0, d_1)$$
$$c \Vdash d_0 : \alpha(x)$$
$$c \Vdash d_1 : \beta(\langle x, d_0 \rangle)\ .$$

$$\begin{tikzcd}
 & \Gamma.\alpha.\beta \arrow[dd, "p_\beta"'] \arrow[dr, equal] & \\
 & & \Gamma.\Sigma_\alpha\beta \arrow[dd, "p_\alpha p_\beta"] \\
 & \Gamma.\alpha \arrow[dr, "p_\alpha"] & \\
\mathrm{y}c \arrow[uuur, bend left, "{\langle\langle x,d_0\rangle,d_1\rangle}"] \arrow[ur, "{\langle x,d_0\rangle}"] \arrow[uurr, bend right, "{\langle x,d\rangle}"'] \arrow[rr, "x"'] & & \Gamma
\end{tikzcd}$$

Forcing dependent product types

Proposition

Given a context Γ, *a type* α *in context* Γ, *a type* β *in context* $\Gamma.\alpha$, *an object* c *of* $\mathcal{C}$, *and a morphism* $x\colon \mathrm{y}c \to \Gamma$,

$$c \Vdash b : (\Pi_\alpha \beta)(x)$$

iff there is a function b *such for every morphism* $f\colon d \to c$ *in* $\mathcal{C}$, *if*

$$d \Vdash a : \alpha(x.f)$$

then

$$d \Vdash b_f(a) : \beta(\langle x.f, a\rangle)$$

and for every $g\colon d' \to d$, $b_f(a).g = b_{f\circ g}(a.g)$.

$$\begin{tikzcd}
 & & \Gamma.\alpha.\beta \arrow[dd, "\mathrm{p}_\beta"] & \\
 & & & \Gamma.\Pi(\alpha)\,\beta \arrow[dd, "p_{\Pi_\alpha\beta}"] \\
\mathrm{y}d \arrow[uurr, dashed, bend left, "{\langle\langle x.f, a\rangle, b_f(a)\rangle}"] \arrow[rr, "{\langle x.f, a\rangle}"] \arrow[dr, "\mathrm{y}(f)"'] & & \Gamma.\alpha \arrow[dr, "\mathrm{p}_\alpha"] & \\
 & \mathrm{y}c \arrow[rr, "x"'] \arrow[uurr, bend right, "{\langle x, b\rangle}"] & & \Gamma
\end{tikzcd}$$

Forcing for disjunction

Proposition

Let $\varphi, \psi : \Gamma \to \Omega$. *For* $x : \mathrm{y}(c) \to \Gamma$, *the following conditions are equivalent:*

1. $c \Vdash \varphi(x) \vee \psi(x)$.
2. *There exists* $u : \mathrm{y}(c) \to E$ *such that* $c \Vdash u : \{\varphi\} + \{\psi\}(x)$.
3. $c \Vdash \varphi(x)$ *or* $c \Vdash \psi(x)$.

Forcing for existential quantifiers

Proposition

Let $\alpha\colon \Gamma \to U$ *and* $\varphi\colon \Gamma.\alpha \to \Omega$. *For* $x\colon \mathrm{y}(c) \to \Gamma$, *the following are equivalent:*

1. $c \Vdash (\exists_\alpha \varphi)(x)$
2. *There exists* $a\colon \mathrm{y}(c) \to E$ *such that* $c \Vdash a : \alpha(x)$, *and* $c \Vdash \varphi(x, a)$.

3. Cofibrations and uniform trivial fibrations
    - The universe of cofibrant propositions
    - The type of partial elements
    - The type TFib
    - *Uniform* trivial fibration structure via Kripke–Joyal semantics

The universe Φ of cofibrant propositions (I)

- As in (Orton and Pitts, 2018), we consider a modality $\mathsf{cof} : \Omega \to \Omega$ satisfying:
  (i) $\mathsf{cof} \circ \mathsf{true} = \mathsf{true}$,
  (ii) $\mathsf{cof} \circ \mathsf{false} = \mathsf{true}$,
  (iii) $\forall(\varphi, \psi : \Omega).\ \mathsf{cof}\,\varphi \Rightarrow (\varphi \Rightarrow \mathsf{cof}\,\psi) \Rightarrow \mathsf{cof}(\varphi \wedge \psi)$.
- The last axiom is called the **principle of dominance** (Rosolini, 1986).

The universe Φ of cofibrant propositions (II)

- Obtain $m_\Phi \colon \Phi \rightarrowtail \Omega$ as the comprehension subtype; in the internal language

$$\Phi \triangleq \{\varphi \in \Omega \mid \mathsf{cof}\ \varphi\}$$

$$\begin{tikzcd} \Phi \arrow[r] \arrow[d, tail, "m_{\mathsf{cof}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\mathsf{true}"] \\ \Omega \arrow[r, "\mathsf{cof}"'] & \Omega \end{tikzcd}$$

- $\mathsf{cof}\,(\mathsf{true}) = \mathsf{true}$ implies that $\mathsf{true} = m_\Phi \circ \mathsf{t}$ for a monomorphism $\mathsf{t} \colon 1 \rightarrowtail \Phi$.

$$\begin{tikzcd} 1 \arrow[r, tail, "\mathsf{t}"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi \arrow[r] \arrow[d, tail, "m_\Phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\mathsf{true}"] \\ 1 \arrow[r, tail, "\mathsf{true}"'] & \Omega \arrow[r, "\mathsf{cof}"'] & \Omega \end{tikzcd}$$

- Call t the **generic cofibrant proposition**.

Forcing Φ

Proposition

Let $\varphi \colon \Gamma \to \Omega$ be a proposition. For every $x \colon \mathrm{y}(c) \to \Gamma$, the following conditions are equivalent.

1. $c \Vdash \mathsf{cof}\,\varphi(x)$
2. $\varphi(x) \colon \mathrm{y}(c) \to \Omega$ *factors through* $\Phi \rightarrowtail \Omega$*.*

$$
\begin{tikzcd}
 & {[\mathsf{cof}\,\varphi]} \arrow[r] \arrow[d, tail] & \Phi \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "\mathsf{true}"] \\
\mathrm{y}(c) \arrow[r, "x"'] \arrow[ur, dotted] & \Gamma \arrow[r, "\varphi"'] & \Omega \arrow[r, "\mathsf{cof}"'] & \Omega
\end{tikzcd}
$$

Cofibrations

Definition

A monomorphism $m \colon C \rightarrowtail Z$ is a **cofibration** if its classifying map $\chi_m \colon Z \to \Omega$ factors through $m_{\mathrm{cof}} \colon \Phi \rightarrowtail \Omega$.

$$
\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[r] \arrow[d, tail, "\mathrm{t}"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\mathrm{true}"] \\
Z \arrow[r, dashed] \arrow[rr, bend right, "\chi_m"'] & \Phi \arrow[r, tail, "m_{\mathrm{cof}}"] & \Omega
\end{tikzcd}
$$

Therefore, all cofibrations are the pullbacks of the generic cofibration $\mathrm{t} \colon 1 \rightarrowtail \Phi$.

Proposition

$m \colon C \rightarrowtail Z$ is a cofibration $\Leftrightarrow$ $\mathscr{E} \Vdash \forall z : Z.\, \mathrm{cof}(\exists c : C.\, m(c) = z)$.

Forcing dominance

Proposition

The following statements are equivalent.

1. *Cofibrations are closed under composition.*
2. $\mathscr{E} \models (\forall \varphi, \psi : \Omega)\,\mathsf{cof}\,\varphi \Rightarrow (\varphi \Rightarrow \mathsf{cof}\,\psi) \Rightarrow \mathsf{cof}(\varphi \wedge \psi)$.
3. *There is a unique dotted arrow (in the top row) making the top square commute.*

$$\begin{tikzcd}
(\Sigma\varphi : \Phi)\Phi^{\{\varphi\}} \arrow[r, dotted, "\exists"] \arrow[d, tail] & \Phi \arrow[d, tail] \\
(\Sigma\varphi : \Omega)\Omega^{\{\varphi\}} \arrow[r, "\exists"] \arrow[d, tail] & \Omega \arrow[d, tail] \\
(\Sigma\alpha : U)U^{\alpha} \arrow[r, "\Sigma"'] & U
\end{tikzcd}$$

The type of partial elements

We define $(-)^+ \colon \mathscr{E} \to \mathscr{E}$ to be the polynomial endofunctor associated to the map $\mathsf{t} \colon 1 \to \Phi$, namely the composite

$$\mathscr{E} \xrightarrow{\mathsf{t}_*} \mathscr{E}_{/\Phi} \xrightarrow{\Phi_!} \mathscr{E}\,.$$

If A is classified by α, then A^+ is classified by

$$\alpha^+ := \textstyle\sum_{\varphi \,:\, \Phi} \{\varphi\} \to \alpha\,,$$

We call α^+ the **type of cofibrant partial elements** of a type α

Forcing for partial elements

Proposition (Forcing for partial elements)

Let $\alpha : \Gamma \to U$ *and* $x : \mathrm{y}(c) \to \Gamma$. *Then the following conditions are equivalent.*

1. $c \Vdash (\varphi, u) : \alpha^+(x)$
2. $c \Vdash \varphi(x) : \Phi$ *and for every* $f : d \to c$, *if* $d \Vdash p : \{\varphi\}(xf)$ *then* $d \Vdash \mathsf{app}(u_f, p) : \alpha(xf)$, *and furthermore the following uniformity condition holds:*

$$\mathsf{app}(u_f, p)g = \mathsf{app}(u_{fg}, p)$$

for any $g : e \to d$ *in* $\mathcal{C}$.

Uniform trivial fibrations

Recall that the class $\mathcal{C}$ of cofibrations determines an awfs $(\mathcal{C}, \mathcal{TF})$ where the right class $\mathcal{TF}$ consists of uniform trivial fibrations.

Uniform trivial fibrations

A **uniform trivial fibration structure** on a small map $p \colon A \to \Gamma$ assigns to every cofibration $C \rightarrowtail Z$ and to every commutative square a diagonal filler $j_C(z,a) \colon Z \to A$, subject to the following *uniformity* condition: for any map $f \colon Z' \to Z$, giving rise to the pullback square on the left, we have

$$j_{C'}(zf, af') = j_C(z,a) \circ f\,.$$

$$\begin{tikzcd}
C' \arrow[r, "f'"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "a"] \arrow[d, tail] & A \arrow[d, "p"] \\
Z' \arrow[r, "f"'] \arrow[urr, dotted, "j_{C'}(zf,af')"] & Z \arrow[r, "z"'] \arrow[ur, dotted, "j_C(z,a)"] & \Gamma
\end{tikzcd}$$

Uniform trivial fibration structure via Kripke–Joyal semantics (I)

For any type α define

$$\mathrm{TFib}(\alpha) := \prod_{\varphi:\Phi} \prod_{u:\{\varphi\}\to\alpha} \sum_{a:\alpha} u =^{\varphi} a \,,$$

where the type $u =^{\varphi} a$ is defined

$$(u =^{\varphi} a) := \prod_{p:\{\varphi\}} \mathrm{app}(u, p) =_{\alpha} a \,.$$

Proposition

The map $p_\alpha : \Gamma.\alpha \to \Gamma$ is a uniform trivial fibration $\Leftrightarrow$ there is a term $\Gamma \vdash t : \mathrm{TFib}(\alpha)$.

Uniform trivial fibration structure via Kripke–Joyal semantics (II)

Lemma

For $\alpha\colon \Gamma \to U$, $x\colon \mathrm{y}c \to \Gamma$ *such that*

$$c \Vdash a : \alpha(x)$$
$$c \Vdash \varphi : \Phi$$
$$c \Vdash u : \big(\{\varphi\} \to \alpha\big)(x)\,.$$

then we also have

$$c \Vdash e\colon (u =^{\varphi} a)(x) \qquad \Leftrightarrow \qquad \begin{tikzcd} {[\varphi]} \arrow[r, "u"] \arrow[d, tail] & \Gamma.\alpha \arrow[d, "p_\alpha"] \\ \mathrm{y}c \arrow[ur, "{\langle x,a \rangle}" description] \arrow[r, "x"'] & \Gamma \end{tikzcd} \quad \textit{commutes.}$$

Proof of Theorem.

Suppose $\Gamma \vdash t : \mathsf{TFib}(\alpha)$. Thus for all $x \colon \mathrm{y}c \to \Gamma$, we have $c \Vdash t_x : \mathsf{TFib}(\alpha)(x)$, coherently in x.

Note that

$$\begin{aligned} \mathsf{TFib}(\alpha) &= \prod_{\varphi:\Phi} \prod_{u:[\varphi]\to\alpha} \sum_{a:\alpha} (u =^{\varphi} a) \\ &= \prod_{(\varphi,u)\,:\,\alpha^+} \sum_{a:\alpha} u =^{\varphi} a \end{aligned}$$

We thus obtain

$$c \Vdash t_x : \prod_{(\varphi,u)\,:\,\alpha^+} \sum_{a:\alpha} (u =^{\varphi} a)(x) \ .$$

Uniform trivial fibration structure via Kripke–Joyal semantics (IV)

Proof of Theorem (cont'd).

By Kripke–Joyal semantics of $\prod$ and $\sum$, we have for every $f \colon d \to c$ in $\mathcal{C}$, if

$$d \Vdash (\varphi, u) : \alpha^{+}(x.f)$$

then

$$d \Vdash t_{x.f}(\varphi, u)^0 : \alpha(x.f)$$

and

$$d \Vdash t_{x.f}(\varphi, u)^1 : \big(u =_{\varphi} t_{x.f}(\varphi, u)^0\big)(x.f)$$

and, for any $g \colon d' \to d$,

$$t_{x.f}(\varphi, u).g = t_{(x.fg)}(\varphi.g, u.g)\,.$$

$$\begin{tikzcd}
{[\varphi.f]} \arrow[r, "{\langle x.f, u_f \rangle}"] \arrow[d, tail] & \Gamma.\alpha \arrow[d, "p_\alpha"] \\
\mathrm{y}d \arrow[r, "x.f"'] & \Gamma
\end{tikzcd}$$

$$\begin{tikzcd}
{[\varphi.f]} \arrow[r, "{\langle x.f, u_f \rangle}"] \arrow[d, tail] & \Gamma.\alpha \arrow[d, "p_\alpha"] \\
\mathrm{y}d \arrow[r, "x.f"'] \arrow[ur, "{t_{x.f}(\varphi,u)^0}" description] & \Gamma
\end{tikzcd}$$

Thus forcing $\mathsf{TFib}(\alpha)$ produces diagonal fillers

$$j_\varphi(x,u) \triangleq t_{x.\,\mathrm{id}_c}(\varphi,u)^0$$

for each lifting problem as in the right hand square below:

$$\begin{tikzcd}
{[\varphi.f]} \arrow[r] \arrow[dd, tail] \arrow[dr, phantom, "\lrcorner", very near start] & {[\varphi]} \arrow[rr, "\langle x,u\rangle"] \arrow[dd, tail] \arrow[dr] & & \Gamma.\alpha \arrow[dd, "p_\alpha"] \\
& & 1 \arrow[dd, "\mathrm{t}"] & \\
\mathrm{y}d \arrow[r, "\mathrm{y}f"'] \arrow[uurrr, dashed, "t_{x.f}(\varphi.f,u.f)^0"] \arrow[drr, bend right, "\varphi.f"'] & \mathrm{y}c \arrow[rr, "x"'] \arrow[uurr, dashed, "t_x(\varphi,u)^0"] \arrow[dr, "\varphi"'] & & \Gamma \\
& & \Phi &
\end{tikzcd}$$

Uniform trivial fibration structure via Kripke–Joyal semantics (VI)

Proof of Theorem (cont'd) – the converse argument

If $p_\alpha : \Gamma.\alpha \to \Gamma$ is a uniform trivial fibration then in particular for every *basic* cofibration $[\varphi] \rightarrowtail \mathrm{y}c$ and square as on the right below, there is a diagonal filler $j_\varphi(x,u)$ as indicated.

$$\begin{tikzcd}
{[\varphi.f]} \arrow[r] \arrow[d, tail] & {[\varphi]} \arrow[r, "u"] \arrow[d, tail] & \Gamma.\alpha \arrow[d, "p_\alpha"] \\
\mathrm{y}c' \arrow[r, "\mathrm{y}f"'] \arrow[urr, dotted, "j_{\varphi.f}(x.f,u.f)"] & \mathrm{y}c \arrow[r, "x"'] \arrow[ur, dotted, "j_\varphi(x,u)"'] & \Gamma
\end{tikzcd}$$

By the lemma, this corresponds to an element $t_x : \mathrm{y}c \to \mathrm{TFib}(\alpha)$ over $x : \mathrm{y}c \to \Gamma$,

$$\begin{tikzcd}
& \Gamma.\mathrm{TFib}(\alpha) \arrow[d, "p_{\mathrm{TFib}(\alpha)}"] \\
\mathrm{y}c \arrow[r, "x"'] \arrow[ur, dotted, "t_x"] & \Gamma
\end{tikzcd}$$

The uniformity condition says exactly that for all $f : c' \to c$, the elements t_x cohere, $t_{(x.f)} = t_x \circ \mathrm{y}(f)$. By Yoneda for the slice category $\mathscr{E}/\Gamma$ that there is a term $\Gamma \vdash \alpha : \mathrm{TFib}(\alpha)$. QED.

4. Trivial cofibrations and uniform fibrations
    - A tiny interval
    - *Uniform* fibration structures from forcing for type Fib
    - Universal uniform fibration

The interval (I)

An **interval with connections** is a presheaf $\mathbb{I}$ in $\mathscr{E}$ equipped with endpoints, i.e. maps $\delta^k \colon 1 \to \mathbb{I}$, for $k \in \{0,1\}$, and connections, i.e. maps $c_k \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ for $k \in \{0,1\}$, satisfying the following axioms.

1. $\delta^0 \neq \delta^1$
2. $\delta^k \colon 1 \to \mathbb{I}$ is a cofibration, for $k \in \{0,1\}$.
3. The diagrams

$$\begin{tikzcd} \mathbb{I} \arrow[r, "{(\delta^k,1)}"] \arrow[d] & \mathbb{I} \times \mathbb{I} \arrow[d, "c_k"] & \mathbb{I} \arrow[l, "{(1,\delta^k)}"'] \arrow[d] \\ 1 \arrow[r, "\delta^k"'] & \mathbb{I} & 1 \arrow[l, "\delta^k"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{I} \arrow[r, "{(\delta^{1-k},1)}"] \arrow[dr, equal] & \mathbb{I} \times \mathbb{I} \arrow[d] & \mathbb{I} \arrow[l, "{(1,\delta^{1-k})}"'] \arrow[dl, equal] \\ & \mathbb{I} & \end{tikzcd}$$

commute, for $k \in \{0,1\}$.

The interval (II)

By Kripke–Joyal semantics the diagrammatic axioms (1-3) above are equivalent to the axioms (1-4) in below written in the language $\tau(\mathscr{E})$:

1. $\neg(0 = 1)$.
2. $(\forall i : \mathbb{I})\big(\, \mathrm{cof}(i = 0) \ \wedge \ \mathrm{cof}(i = 1)\big)$.
3. $(\forall i : \mathbb{I})\big(c_0(0, i) = 0 = c_0(i, 0) \wedge c_0(1, i) = i = c_0(i, 1)\big)$.
4. $(\forall i : \mathbb{I})\big(c_1(0, i) = i = c_1(i, 0) \wedge c_1(i, 1) = 1 = c_1(1, i)\big)$.

Naive trivial cofibrations

A **naive trivial cofibrations** is a Leibniz tensors of the form

$$c \hat{\otimes} \delta_k : Z +_C (Z \times \mathbb{I}) \rightarrowtail Z \times \mathbb{I}\,,$$

for an arbitrary cofibration $c\colon C \rightarrowtail Z$ and an endpoint $\delta_k\colon 1 \rightarrowtail \mathbb{I}$, for $k \in \{0,1\}$. This construction is stable under pullback, in the sense that for any map $t\colon Z' \to Z$, one has a pullback square

$$\begin{tikzcd}
Z' +_{C'} (C' \times \mathbb{I}) \arrow[r] \arrow[d, tail, "(t^* c)\hat{\otimes}\delta_k"'] & Z +_C (C \times \mathbb{I}) \arrow[d, tail, "c\hat{\otimes}\delta_k"] \\
Z' \times \mathbb{I} \arrow[r, "h\times\mathbb{I}"'] & Z \times \mathbb{I}
\end{tikzcd}$$

Uniform fibration structure

A **uniform fibration structure** on a small map $p \colon A \to \Gamma$ consists of a function j that assigns a dotted filler $j(i, u, v) \colon Z \times \mathbb{I} \to A$ to every diagram of solid arrows

$$\begin{tikzcd}
Z +_C (C \times \mathbb{I}) \arrow[r, "u"] \arrow[d, tail, "c \otimes \delta_k"'] & A \arrow[d] \\
Z \times \mathbb{I} \arrow[r, "v"'] & \Gamma
\end{tikzcd}$$

where c is a cofibration and $k \in \{0, 1\}$, subject to the following *uniformity condition*: for any map $t \colon Z' \to Z$ and induced pullback square on the left,

$$\begin{tikzcd}
Z +'_C (C' \times \mathbb{I}) \arrow[r] \arrow[d, tail, "c' \otimes \delta"'] & Z +_C (C \times \mathbb{I}) \arrow[r, "u"] \arrow[d, tail] & A \arrow[d] \\
Z' \times \mathbb{I} \arrow[r, "h \times \mathbb{I}"'] \arrow[urr, dotted] & Z \times \mathbb{I} \arrow[r, "v"'] \arrow[ur, dotted] & \Gamma,
\end{tikzcd} \tag{1}$$

we have that $j(i, uh', vh) = j(i, u, v) \circ (h \times \mathbb{I})$.

The type of fillings

Let $\alpha : \mathbb{I} \to U$.

Recall the **type of 0-directed filling structure**

$$\mathsf{Fill}_0(\alpha) = \prod_{\varphi : \Phi} \prod_{u : \{\varphi\} \to \prod_{i : \mathbb{I}} \alpha(i)} \prod_{a : \alpha_0} (\tilde{u}_0 =^{\varphi} a) \to \sum_{s : \prod_{i : \mathbb{I}} \alpha(i)} (s_0 =_{\alpha_0} a) \times (u =^{\varphi} s)\,,$$

where

$$(\tilde{-}) \colon \left(\{\varphi\} \to \textstyle\prod_{i : \mathbb{I}} \alpha(i)\right) \;\cong\; \textstyle\prod_{i : \mathbb{I}} \{\varphi\} \to \alpha(i)$$

For any type $\Gamma \colon U$ and family of types $\alpha \colon \Gamma \to U$, we then define the **type of fibration structures**

$$\mathsf{Fib}(\alpha) := \prod_{x : \mathbb{I} \to \Gamma} \mathsf{Fill}_0(\alpha(x)) \times \mathsf{Fill}_1(\alpha(x))\,.$$

Forcing for type Fib

By Kripke-Joyal semantics, we have:

Theorem

Let $\alpha : \Gamma \to U$. Then the following conditions are equivalent.

1. *The map $p_\alpha : \Gamma.\alpha \to \Gamma$ is a uniform fibration.*
2. *There is a term $t : \mathrm{Fib}(\alpha)$.*

Universal uniform fibration (I)

Unfortunately the type

$$\mathrm{Fib}(\alpha) := \prod_{x\,:\,\mathbb{I}\to\Gamma} \mathrm{Fill}_0(\alpha(x)) \times \mathrm{Fill}_1(\alpha(x))\,.$$

is not indexed over Γ and is not good candidate for the construction universal uniform fibration.

Instead, we need the further assumption that the interval $\mathbb{I}$ is **tiny**:

$$(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$$

Using the amazing right adjoint, we define the universal fibration $\mathrm{Fib}^* \to U$ as the pullback of $E \to U$ along $(\mathrm{Fill})_{\mathbb{I}} \circ \eta$.

$$\begin{tikzcd}
\mathrm{Fib}^*(\alpha) \arrow[r] \arrow[d] & \mathrm{Fib}^* \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & (E)_{\mathbb{I}} \arrow[d] \\
\Gamma \arrow[r, "\alpha"'] \arrow[u, dashed, bend left] & U \arrow[r, "\eta"'] & (U^{\mathbb{I}})_{\mathbb{I}} \arrow[r, "(\mathrm{Fill})_{\mathbb{I}}"'] & (U)_{\mathbb{I}}
\end{tikzcd}$$

Universal uniform fibration (II)

Theorem

Let $\alpha\colon \Gamma \to U$.

1. *There is a bijection between points $1 \to \mathsf{Fib}(\alpha)$ and sections of $\mathsf{Fib}^*(\alpha)$ over Γ, which is natural in Γ.*
2. *The object $\mathsf{Fib}^*(\alpha)$ is stable under pullback along any map $\gamma\colon \Delta \to \Gamma$.*

$$\begin{tikzcd}
\Delta.\mathsf{Fib}^*(\gamma(t)) \arrow[r] \arrow[d] & \Gamma.\mathsf{Fib}^*(\alpha) \arrow[d] \\
\Delta \arrow[r, "\gamma"'] & \Gamma \\
\Delta.\alpha(\gamma) \arrow[r] \arrow[u] & \Gamma.\alpha \arrow[u]
\end{tikzcd}$$

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The End

Thanks for your attention!