

# Kripke-Joyal Semantics for Dependent Type Theory<sup>1</sup>

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# Outline

- 1 Review of classical Kripke-Joyal semantics for toposes
- 2 Kripke–Joyal semantics for dependent type theories
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  - Forcing dependent sum types
  - Forcing dependent product types
  - Forcing universe of small types
  - Forcing the impredicative universe  $\Omega$  of propositions
  - Forcing the type of partial elements
  - Applications

## Review of classical Kripke-Joyal semantics for toposes

- ▶ **The Kripke–Joyal semantics** of a topos  $\mathcal{E}$  gives an interpretation to formulas written in its higher order intuitionistic internal language  $HoL(\Sigma_{\mathcal{E}})$ .

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- ▶ The Kripke–Joyal semantics is in fact a higher order generalization of the well-known Kripke semantic for intuitionistic propositional logic.

## Definition

Let  $\mathcal{E}$  be an elementary topos. Given a formula  $\varphi(x)$  with a free variable  $x$  of sort  $A$  in  $HoL(\Sigma_{\mathcal{E}})$ , and a generalized element  $\alpha: U \rightarrow A$  in  $\mathcal{E}$ , we define

$$U \Vdash \varphi(\alpha) \Leftrightarrow \alpha \text{ factors through the subobject } [\varphi] \rightarrow A.$$

$$\begin{array}{ccccc} & & [\varphi] & \xrightarrow{!} & 1 \\ & \nearrow & \downarrow & \lrcorner & \downarrow \text{true} \\ U & \xrightarrow{\alpha} & X & \xrightarrow{\varphi} & \Omega \end{array}$$

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- ▶ Call  $U$  the **stage** of forcing.
- ▶ Write  $\mathcal{E} \Vdash \varphi$  if at every stage  $U$  and for every generalized element  $\alpha$ , we have  $U \Vdash \varphi(\alpha)$ .

One can then show:

- ▶  $U \Vdash \top(\alpha)$ .
- ▶  $U \Vdash \perp(\alpha)$  iff  $U$  is the initial object of  $\mathcal{E}$ .
- ▶  $U \Vdash (x = x')(\langle \alpha, \alpha' \rangle)$  iff  $\alpha: U \rightarrow X$  and  $\alpha': U \rightarrow X$  are the same maps in  $\mathcal{E}$ .
- ▶  $U \Vdash (\varphi \wedge \psi)(\alpha)$  iff  $U \Vdash \varphi(\alpha)$  and  $U \Vdash \psi(\alpha)$ .
- ▶  $U \Vdash (\varphi \vee \psi)(\alpha)$  iff there are jointly epimorphic arrows  $p: V \rightarrow U$  and  $q: W \rightarrow U$  such that  $V \Vdash \varphi(\alpha \circ p)$  and  $W \Vdash \psi(\alpha \circ q)$ .
- ▶  $U \Vdash (\varphi \Rightarrow \psi)(\alpha)$  iff for any arrow  $f: V \rightarrow U$  such that  $V \Vdash \varphi(\alpha \circ f)$  then  $V \Vdash \psi(\alpha \circ f)$ .
- ▶  $c \Vdash \neg\varphi(\alpha)$  iff for all maps  $f: V \rightarrow U$  in  $\mathcal{E}$ ,  $V \nVdash \varphi(\alpha \circ f)$ .
- ▶  $\vdots$



► Henceforth,  $\mathcal{E} = \mathcal{P}\text{Shv}(\mathcal{C}) = \text{Set}^{\mathcal{C}^{\text{op}}}$ .

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- ▶ So it is enough to consider forcing statements  $U \Vdash \varphi(\alpha)$  for representables  $U = y_{\mathcal{C}}$ .

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- ▶ So it is enough to consider forcing statements  $U \Vdash \varphi(\alpha)$  for representables  $U = y c$ .

$$c \Vdash (\varphi \vee \psi)(\alpha) \Leftrightarrow c \Vdash \varphi(\alpha) \text{ or } c \Vdash \psi(\alpha)$$

Recall  $y c$  is projective & indecomposable.

$$\begin{array}{ccc}
 & & [\varphi] + [\psi] \\
 & \nearrow \text{dashed} & \downarrow \\
 & \xrightarrow{\tilde{\alpha}} & [\varphi] \cup [\psi] \\
 & \searrow \alpha & \downarrow \\
 y c & \xrightarrow{\alpha} & X
 \end{array}$$

# Limitations of classical Kripke–Joyal semantics

- ▶ Bounded quantification. We shall overcome this by generalizing Kripke–Joyal semantics to dependent type theory with universes.
- ▶ Equality of terms is extensional and not “up to homotopy”. We will also generalize to homotopy type theory.

# Kripke–Joyal semantics for dependent type theory

- ▶ Provability versus proof relevance: the classical Kripke–Joyal semantics is only concerned with the provability of a proposition; it is proof irrelevant. Kripke–Joyal semantics for dependent type theory includes the terms (proofs) in the forcing statements.

# Kripke–Joyal semantics for dependent type theory

- ▶ Provability versus proof relevance: the classical Kripke–Joyal semantics is only concerned with the provability of a proposition; it is proof irrelevant. Kripke–Joyal semantics for dependent type theory includes the terms (proofs) in the forcing statements.
- ▶ We want a sound, formal and (*quasi-*) *mechanical* process to relate internal developments (Cohen et al., 2018), (Orton and Pitts, 2018), etc. with the diagrammatic developments (Gambino and Sattler, 2017), (Sattler, 2017), (Awodey, 2018), etc. found in the models of HoTT literature.

## Kripke–Joyal semantics for dependent type theories

- ▶ We work in ZFC extended with two inaccessible cardinals  $\kappa < \lambda$ .



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- ▶ We fix a  $\lambda$ -small category  $\mathcal{C}$ . We define the Grothendieck topos of presheaves

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- ▶ As usual, we get classifiers  $\pi: \text{Type}^\bullet \rightarrow \text{Type}$  and  $\pi_{\mathcal{V}}: \mathcal{V}^\bullet \rightarrow \mathcal{V}$  for  $\lambda$ -small and  $\kappa$ -small families, resp.

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Following (Awodey, 2018), we get a CwF structure on  $\mathcal{E}$  from the universe  $\pi: \mathbf{Type}^\bullet \rightarrow \mathbf{Type}$ :

Steve Awodey. “Natural models of homotopy type theory”. In: *Math. Structures Comput. Sci.* 28.2 (2018)

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A **term**  $a: A$  in context  $\Gamma$  is a map  $a: \Gamma \rightarrow \text{Type}^\bullet$  with  $\pi \circ a = A$ .

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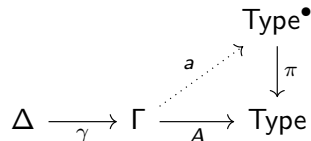
The **context extension** is given by the pullback along

$\pi$ .

$$\begin{array}{ccc} \Gamma.A & \longrightarrow & \text{Type}^\bullet \\ \downarrow \Gamma & & \downarrow \pi \\ \Gamma & \xrightarrow{A} & \text{Type} \end{array}$$

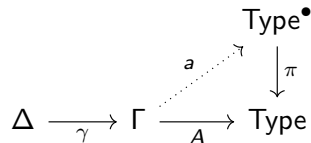
$$\frac{A \in \text{Type}(\Gamma) \quad \gamma: \Delta \rightarrow \Gamma}{A(\gamma) \triangleq A \circ \gamma \in \text{Type}(\Delta)} \text{ (substitution-types)}$$

$$\frac{a \in \text{Term}(\Gamma, A) \quad \gamma: \Delta \rightarrow \Gamma}{a(\gamma) \triangleq a \circ \gamma \in \text{Term}(\Delta, A(\gamma))} \text{ (substitution-terms)}$$



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When  $\Delta = y c$ :

by the Yoneda lemma,  $\gamma \in \Gamma(c)$ , and  $A(\gamma) \in \text{Type}(c)$ .

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$$\begin{array}{ccccc} & & & & \text{Type}^\bullet \\ & & & & \downarrow \pi \\ \Delta & \xrightarrow{\gamma} & \Gamma & \xrightarrow{A} & \text{Type} \\ & & \nearrow a & & \end{array}$$

When  $\Delta = y c$ :

by the Yoneda lemma,  $\gamma \in \Gamma(c)$ , and  $A(\gamma) \in \text{Type}(c)$ .

For a morphism  $f: d \rightarrow c$  in  $\mathcal{C}$ , we write  $\gamma.f$  for the composite  $\gamma \circ y f$ .

Also, for  $a \in \text{Term}(y c, A(\gamma))$  and we have  $a(\gamma.f) = a(\gamma).f \in \text{Term}(d, A(\gamma.f))$ .

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$\mathcal{T}(\mathcal{E})$  has

- ▶ the basic types  $\emptyset$ ,  $\mathbf{1}$ ,  $\mathbf{2}$ ,  $N$ .
- ▶ also the following forms of type:

$$A \times B, \quad A + B, \quad A \rightarrow B$$
$$\text{Eq}_A(a, b), \quad \sum_{x:A} B(x), \quad \prod_{x:A} B(x)$$

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$$\text{Eq}_A(a, b), \quad \sum_{x:A} B(x), \quad \prod_{x:A} B(x)$$

- ▶ The type of extensional equality is given by the diagonal map  $\delta: A \rightarrow A \times A$ .
- ▶ These types satisfy the usual induction and computation rules, e.g. in HoTT-Book.
- ▶ There is an evident tautological interpretation of  $\mathcal{T}(\mathcal{E})$  into  $\mathcal{E}$ , using the CwF structure of  $\mathcal{E}$ .

$\mathcal{T}(\mathcal{E})$  additionally has the types

- ▶  $\mathcal{V}$ : the universe of small types

$$\frac{a \in \text{Term}(\Gamma, \mathcal{V})}{\text{El}(a) \in \text{Term}(\Gamma, \text{Type})}$$

$$\begin{array}{ccccc} \Gamma.\text{El}(a) & \longrightarrow & \mathcal{V}^\bullet & \xrightarrow{\gamma} & \text{Type}^\bullet \\ \downarrow \ulcorner & & \downarrow \ulcorner & & \downarrow \\ \Gamma & \xrightarrow{a} & \mathcal{V} & \xrightarrow{\text{El}} & \text{Type} \end{array}$$

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- ▶  $\mathcal{V}$ : the universe of small types
- ▶  $\Omega$ : the impredicative universe of propositions.

$$\frac{\varphi \in \text{Term}(\Gamma, \omega)}{\iota(\varphi) \in \text{Term}(\Gamma, \text{Type})}$$

$$\begin{array}{ccccc} [\varphi] & \longrightarrow & 1 & \xrightarrow{\tilde{*}} & \text{Type}^\bullet \\ \downarrow \Gamma & & \downarrow \Gamma_{\text{true}} & & \downarrow \pi \\ \Gamma & \xrightarrow{\varphi} & \Omega & \xrightarrow{\iota} & \text{Type} \end{array}$$

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- ▶  $\mathcal{V}$ : the universe of small types
- ▶  $\Omega$ : the impredicative universe of propositions.
- ▶ Note that

$$\text{El}(\iota(\varphi)) \cong [\varphi]$$

As in (Orton and Pitts, 2018), we consider a modality  $\text{cof} : \Omega \rightarrow \Omega$  satisfying:

- (i)  $\text{cof} \circ \text{true} = \text{true}$ ,
- (ii)  $\text{cof} \circ \text{false} = \text{true}$ ,
- (iii)  $\forall(\varphi, \psi : \Omega). \text{cof } \varphi \Rightarrow (\varphi \Rightarrow \text{cof } \psi) \Rightarrow \text{cof}(\varphi \wedge \psi)$ .

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The last axiom is called the **principle of dominance**.

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Obtain  $m_{\text{Cof}} : \text{Cof} \rightarrow \Omega$  as the comprehension subtype.

$$\text{Cof} \triangleq \{\varphi \in \Omega \mid \text{cof } \varphi\}$$

$$\begin{array}{ccccc}
 1 & \xrightarrow{\text{t}} & \text{Cof} & \longrightarrow & 1 \\
 \lrcorner & & \downarrow m_{\text{Cof}} & \lrcorner & \downarrow \text{true} \\
 1 & \xrightarrow{\text{true}} & \Omega & \xrightarrow{\text{cof}} & \Omega
 \end{array}$$



# Dependent Kripke–Joyal semantics

Definition (Dependent Kripke–Joyal semantics– forcing terms)

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For a context  $\Gamma$ , a type  $\Gamma \vdash A \text{ Type}$ , an object  $c$  of  $\mathcal{C}$ , and a morphism  $\gamma: yc \rightarrow \Gamma$ ,

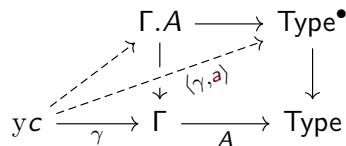
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$c \Vdash a : A(\gamma) \Leftrightarrow$  there is a lift  $\langle \gamma, a \rangle$  of  $\gamma$  against  $p_A: \Gamma.A \rightarrow \Gamma$ .

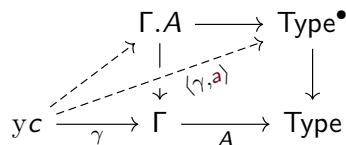


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Proposition

$\Gamma \vdash a : A \Leftrightarrow$  There is a family  $(a_\gamma \mid c : \text{an object of } \mathcal{C}, \gamma: yc \rightarrow \Gamma)$  satisfying

$$c \Vdash a_\gamma : A(\gamma)$$

and for every morphism  $f: d \rightarrow c$  of  $\mathcal{C}$ ,

$$a_\gamma \cdot f = a_\gamma \cdot f$$



# Forcing extensional equality types

## Proposition

Given a context  $\Gamma$ , a type  $\Gamma \vdash A$  Type,

$$\begin{array}{c} \Gamma.A \\ \downarrow \delta \\ A \times_{\Gamma} A \\ \downarrow \\ \Gamma \end{array}$$



# Forcing extensional equality types

## Proposition

Given a context  $\Gamma$ , a type  $\Gamma \vdash A$  Type, an object  $c$  of  $\mathcal{C}$ ,

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$yc$

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## Forcing extensional equality types

### Proposition

Given a context  $\Gamma$ , a type  $\Gamma \vdash A$  Type, an object  $c$  of  $\mathcal{C}$ , a morphism  $\gamma: yc \rightarrow \Gamma$ ,  $c \Vdash (a, a') : (A \times A)(\gamma)$

A commutative triangle diagram illustrating the relationship between contexts and types. The vertices are  $yc$  (bottom-left),  $\Gamma$  (bottom-right), and  $A \times_{\Gamma} A$  (top-right). The edges are:

- A horizontal arrow from  $yc$  to  $\Gamma$  labeled  $\gamma$ .
- A diagonal arrow from  $yc$  to  $A \times_{\Gamma} A$  labeled  $\langle \gamma, (a, a') \rangle$ .
- A vertical arrow from  $A \times_{\Gamma} A$  to  $\Gamma$  labeled  $\delta$ .
- A vertical arrow from  $\Gamma.A$  (top) to  $A \times_{\Gamma} A$  labeled  $\delta$ .

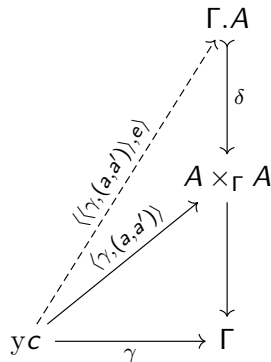
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 $c \Vdash (a, a') : (A \times A)(\gamma)$  we have

$$\begin{aligned} c \Vdash e : \text{Eq}_A(a, a')(\gamma) &\Leftrightarrow \\ a, a' \text{ are equal as morphisms in } \mathcal{E} &\Leftrightarrow \\ a, a' \text{ are equal elements of } A(c) . & \end{aligned}$$

The Type  $\text{Eq}_A$  is interpreted by the diagonal morphism  $\delta: A \rightarrow A \times_{\Gamma} A$  over  $\Gamma$ .

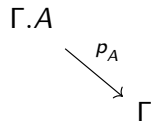




# Forcing dependent sum types

## Proposition

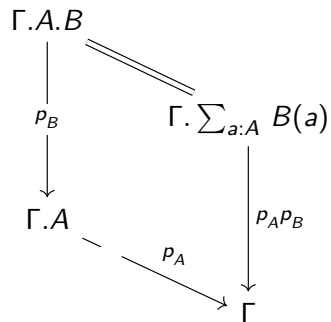
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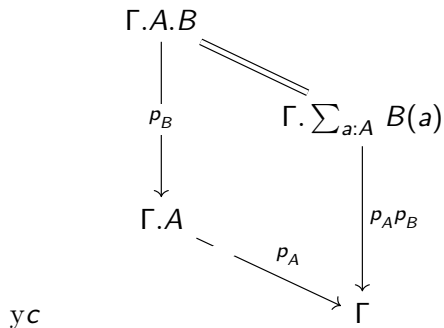
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Given a context  $\Gamma$ , a type  $\Gamma \vdash A$  Type, a type  $\Gamma, x : A \vdash B$  Type, an object  $c$  of  $\mathcal{C}$ ,

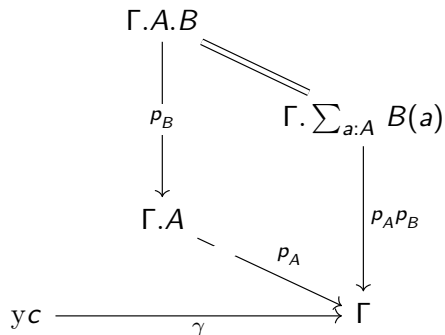




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Given a context  $\Gamma$ , a type  $\Gamma \vdash A$  Type, a type  $\Gamma, x : A \vdash B$  Type, an object  $c$  of  $\mathcal{C}$ , and a morphism  $\gamma : yc \rightarrow \Gamma$ ,



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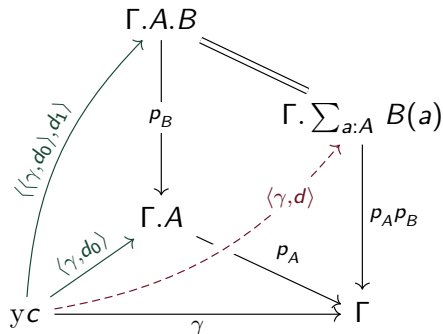
$$c \Vdash d : \left( \sum_{a:A} B(a) \right) (\gamma)$$

iff

$$d = (d_0, d_1)$$

$$c \Vdash d_0 : A(\gamma)$$

$$c \Vdash d_1 : B(\langle \gamma, d_0 \rangle).$$



# Forcing dependent product types

Proposition

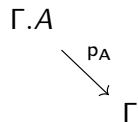
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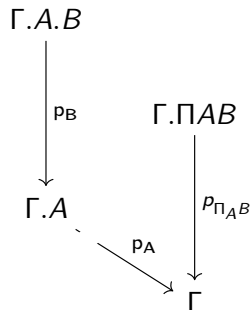
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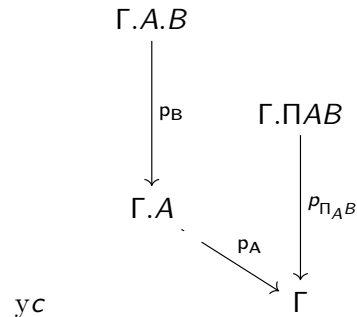
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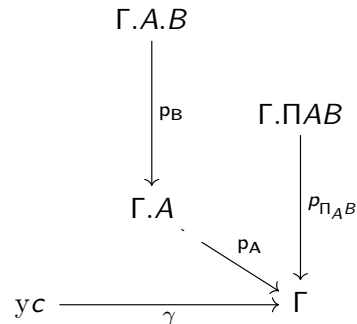
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Given a context  $\Gamma$ , a type  $\Gamma \vdash A$  Type, a type  $\Gamma, x : A \vdash B$  Type, an object  $c$  of  $\mathcal{C}$ , and a morphism  $\gamma : yc \rightarrow \Gamma$ ,

$$c \Vdash b : \left( \prod_{x:A} B \right) (\gamma)$$

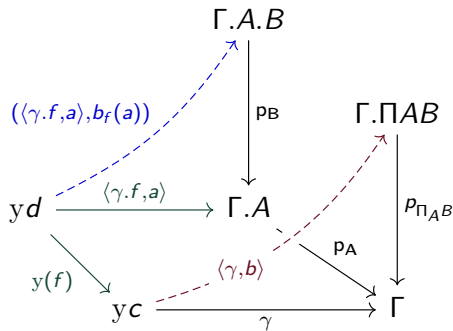
iff there is a function  $b$  such for every morphism  $f : d \rightarrow c$  in  $\mathcal{C}$ , if

$$d \Vdash a : A(\gamma.f)$$

then

$$d \Vdash b_f(a) : B(\langle \gamma.f, a \rangle)$$

and for every  $g : d' \rightarrow d$ ,  $b_f(a).g = b_{f \circ g}(a.g)$ .





## Forcing universe $\mathcal{V}$ of small types

### Proposition

For an object  $c$  of  $\mathcal{C}$ ,

$$c \Vdash a : v \Leftrightarrow c \Vdash \text{El}(a)\text{Type},$$

$\text{El}(a.f) \equiv \text{El}(a).f$  for every  $f : d \rightarrow c$ , and

$\text{El}(a) \rightarrow yc$  and  $\text{El}(a.f) \rightarrow yd$  (for all  $f : d \rightarrow c$ ) are small.

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## Proposition

For an object  $c$  of  $\mathcal{C}$ ,

$$c \Vdash [a^\bullet : \mathcal{V}^\bullet] \Leftrightarrow a^\bullet = (a, b) \text{ such that } c \Vdash a : v$$

$$\text{and } c \Vdash b : \text{El}(a)$$

$$\begin{array}{ccc} \text{El}(a) & \xrightarrow{q_a} & \mathcal{V}^\bullet \\ \begin{array}{c} \uparrow b \\ \downarrow p_a \end{array} \Gamma & & \downarrow p_v \\ yc & \xrightarrow{a} & \mathcal{V} \end{array}$$

# Forcing $\Omega$

## Theorem

Let  $\varphi: \Gamma \rightarrow \Omega$  and  $\gamma: c \rightarrow \Gamma$ . Then the following are equivalent:

- 1  $c \Vdash \varphi(\gamma)$  in the sense of the standard Kripke-Joyal semantics,
- 2 there exists a (necessarily unique)  $a: yc \rightarrow \mathcal{V}^\bullet$  such that  $c \Vdash a: \iota\varphi(\gamma)$ .

$$\begin{array}{ccccccc} & & [\varphi] & \longrightarrow & 1 & \longrightarrow & \text{Type}^\bullet \\ & \nearrow & \downarrow & \Gamma & \downarrow & \text{true} & \downarrow \pi \\ yc & \xrightarrow{\gamma} & \Gamma & \xrightarrow{\varphi} & \Omega & \xrightarrow{\iota} & \text{Type} \end{array}$$

# Cofibrations

## Definition

A monomorphism  $m: C \rightarrow Z$  is a **cofibration** if its classifying map  $\chi_m: Z \rightarrow \Omega$  factors through  $m_{\text{cof}}: \text{Cof} \rightarrow \Omega$ .

$$\begin{array}{ccccc} C & \longrightarrow & 1 & \longrightarrow & 1 \\ m \downarrow \lrcorner & & t \downarrow \lrcorner & & \downarrow \text{true} \\ Z & \dashrightarrow & \text{Cof} & \xrightarrow{m_{\text{cof}}} & \Omega \end{array}$$

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## Proposition

$m: C \rightarrow Z$  is a cofibration  $\Leftrightarrow \mathcal{E} \Vdash \forall z : Z. \text{cof}(\exists c : C. m(c) = z)$ .

Consider the following polynomials

$$1 \xrightarrow{t} \text{Cof}$$

$$1 \xrightarrow{\text{true}} \Omega$$

$$\text{Type} \xrightarrow{P_{\text{Type}}} \text{Type}$$

associated polynomial functor

$$\mathcal{E} \xrightarrow{P_t} \mathcal{E}$$

$$\mathcal{E} \xrightarrow{P_{\text{true}}} \mathcal{E}$$

$$\mathcal{E} \xrightarrow{P_{P_{\text{Type}}}} \mathcal{E}$$

where

$$P_t(A) = \sum_{\varphi: \text{Cof}} A^{[\varphi]}$$

$$P_{\text{true}}(A) = \sum_{\varphi: \Omega} A^{[\varphi]}$$

$$P_{P_{\text{Type}}}(A) = \sum_{a: \text{Type}} A^{\text{El}(a)}$$

Because the square

$$\begin{array}{ccc}
 1 & \longrightarrow & 1 \\
 \downarrow \ulcorner & & \downarrow \text{true} \\
 \text{Cof} & \xrightarrow{m_{\text{cof}}} & \Omega
 \end{array}$$

is cartesian, we obtain a cartesian square:

$$\begin{array}{ccc}
 P_t(\text{Cof}) & \xrightarrow{\quad} & P_{\text{true}}(\text{Cof}) \\
 P_t(m_{\text{cof}}) \downarrow \ulcorner & & \downarrow P_{\text{true}}(m_{\text{cof}}) \\
 P_t(\Omega) & \xrightarrow{\quad} & P_{\text{true}}(\Omega)
 \end{array}
 =
 \begin{array}{ccc}
 \sum_{\varphi:\text{Cof}} \text{Cof}^{[\varphi]} & \xrightarrow{\quad} & \sum_{\varphi:\Omega} \text{Cof}^{[\varphi]} \\
 \downarrow \ulcorner & & \downarrow \\
 \sum_{\varphi:\text{Cof}} \Omega^{[\varphi]} & \xrightarrow{\quad} & \sum_{\varphi:\Omega} \Omega^{[\varphi]}
 \end{array}$$



And because

$$\begin{array}{ccc}
 1 & \xrightarrow{\tilde{*}} & \text{Type}^\bullet \\
 \text{true} \downarrow \ulcorner & & \downarrow P_{\text{Type}} \\
 \Omega & \xrightarrow{\iota} & \text{Type}
 \end{array}$$

is cartesian, we obtain a cartesian square:

$$\begin{array}{ccc}
 P_{\text{true}}(\Omega) & \xrightarrow{\quad} & P_{P_{\text{Type}}}(\Omega) \\
 P_{\text{true}(\iota)} \downarrow \ulcorner & & \downarrow P_{P_{\text{Type}}(\iota)} \\
 P_{\text{true}}(\text{Type}) & \xrightarrow{\quad} & P_{P_{\text{Type}}}(\text{Type})
 \end{array}
 =
 \begin{array}{ccc}
 \sum_{\varphi:\Omega} \Omega^{[\varphi]} & \xrightarrow{\quad} & \sum_{a:\text{Type}} \Omega^{\text{El}(a)} \\
 \downarrow \ulcorner & & \downarrow \\
 \sum_{\varphi:\Omega} \text{Type}^{[\varphi]} & \xrightarrow{\quad} & \sum_{a:\text{Type}} \text{Type}^{\text{El}(a)}
 \end{array}$$

$$\begin{array}{ccccc}
 P_t(\text{Cof}) & \xrightarrow{\quad} & P_{\text{true}}(\text{Cof}) & & \\
 \downarrow & \lrcorner & \downarrow & & \\
 P_t(\Omega) & \xrightarrow{\quad} & P_{\text{true}}(\Omega) & \xrightarrow{\quad} & P_{\rho_{\text{Type}}}(\Omega) \\
 & & \downarrow & \lrcorner & \downarrow \\
 & & P_{\text{true}}(\text{Type}) & \xrightarrow{\quad} & P_{\rho_{\text{Type}}}(\text{Type})
 \end{array}$$

Therefore, there is a composite map

$$\sum_{\varphi: \text{Cof}} \text{Cof}^{[\varphi]} = P_t(\text{Cof}) \xrightarrow{\quad} P_{\text{true}}(\Omega) \xrightarrow{\quad} P_{\rho_{\text{Type}}}(\text{Type}) = \sum_{a: \text{Type}} \text{Type}^{\text{El}(a)}$$

which takes  $(\varphi, \psi)$  to  $(\iota\varphi, \iota\psi)$ .

## Proposition

$\mathcal{E} \Vdash [\text{dom} : \forall(\varphi, \psi : \Omega). \text{cof } \varphi \Rightarrow (\varphi \Rightarrow \text{cof } \psi) \Rightarrow \text{cof}(\varphi \wedge \psi)] \Leftrightarrow$   
*there is a lift dom of  $\Sigma$  making the square commute.*

$$\begin{array}{ccc}
 P_t(\text{Cof}) & \overset{\text{dom}}{\dashrightarrow} & \text{Cof} \\
 \downarrow & & \downarrow \iota \\
 P_{\rho_{\text{Type}}}(\text{Type}) & \xrightarrow{\Sigma} & \text{Type}
 \end{array}$$

- Note that  $\Sigma : P_{\rho_{\text{Type}}}(\text{Type}) \rightarrow (\text{Type})$  in above is the Natural Model (resp. CwF) interpretation of the  $\sum$  type-former following (Awodey, 2018).

## Proposition

For  $\varphi : \mathbf{Cof}$  and  $\psi : [\varphi] \rightarrow \mathbf{Cof}$ , the following statements hold:

- (i)  $\text{dom}(\mathbf{t}, \varphi) = \varphi = \text{dom}(\varphi, \mathbf{t})$ .
- (ii)  $\text{dom}(\text{dom}(\varphi, \psi), \theta) = \text{dom}(\varphi, \text{dom}(\psi, \theta))$ .
- (iii)  $[\text{dom}(\varphi, \psi)] \equiv \sum_{x: [\varphi]} [\psi(x)]$ .

## Proof.

For (i), note that  $\iota(\mathbf{t}) = \text{code}(\mathbf{1})$  where  $\mathbf{1}$  is the terminal type. Since  $\sum_{*: \mathbf{1}} \varphi(*) = \iota\varphi$  and  $\iota$  is monic,  $\text{dom}(\mathbf{t}, \varphi) = \varphi$ .

For (ii), since  $\sum_{x: \iota\varphi} \mathbf{t} \cong \text{code}(\mathbf{1})$  and the "exchange rule" of the sum types.

For (iii), observe that

$$[\text{dom}(\varphi, \psi)] \equiv \text{El}\iota(\text{dom}(\varphi, \psi)) \equiv \text{El}(\Sigma(\iota\varphi, \iota\psi)) \equiv \sum_{x: [\varphi]} [\psi(x)] .$$

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*Cofibrations are closed under composition.*

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## Proof.

It suffices to prove that if  $m_\varphi: [\varphi] \twoheadrightarrow y\mathcal{C}$  and  $m_\psi: [\psi] \twoheadrightarrow [\varphi]$  are cofibrations then so is their composite.

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$c \Vdash \varphi : \text{Cof}$  and  $c \Vdash \psi : [\varphi] \rightarrow \text{Cof}$  imply  $c \Vdash \text{dom}(\varphi, \psi) : \text{Cof}$

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$c \Vdash \varphi : \text{Cof}$  and  $c \Vdash \psi : [\varphi] \rightarrow \text{Cof}$  imply  $c \Vdash \text{dom}(\varphi, \psi) : \text{Cof}$

$\text{dom}(\varphi, \psi)$  classifies  $m_\varphi \circ m_\psi$  since (i)  $[\text{dom}(\varphi, \psi)] \equiv \sum_{x: [\varphi]} [\psi(x)]$ , and (ii)  $m_\varphi \circ m_\psi$  is the display map of the sum type  $\sum_{x: [\varphi]} [\psi(x)]$ .

$$\begin{array}{ccccc}
 [\psi] & \longrightarrow & 1 & & \\
 m_\psi \downarrow \ulcorner & & \downarrow \ulcorner & & \\
 [\varphi] & \xrightarrow[\ulcorner]{\psi} & \text{Cof} & \longrightarrow & 1 \\
 m_\varphi \downarrow & & & & \downarrow \ulcorner \\
 yc & \xrightarrow{\varphi} & \text{Cof} & & \\
 & \searrow \text{---} \ulcorner & & & \\
 & \text{dom}(\varphi, \psi) & & & 
 \end{array}$$



The **type of partial elements** of a type  $A$  is given by the polynomial functor

$$P_{\text{true}}(A) = \sum_{\varphi: \Omega} [\varphi] \rightarrow A.$$

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The type of **cofibrant partial elements** of a type  $A$  is given by the polynomial functor

$$A^+ = P_{\text{t}}(A) = \sum_{\varphi: \text{Cof}} [\varphi] \rightarrow A.$$

There is a natural map

$$\begin{aligned} \eta: A &\longrightarrow A^+ \\ a &\longmapsto (\text{true}, \lambda * . a : 1 \rightarrow A) \end{aligned}$$

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which fits into the pullback square

$$\begin{array}{ccc} A & \xrightarrow{\eta} & A^+ \\ \downarrow !_A & \lrcorner & \downarrow \text{fst} \\ 1 & \xrightarrow{\tau} & \text{Cof} \end{array} \cdot$$

**Proposition (Awodey,2018)**

*The map  $\eta_A: A \rightarrow A^+$  is a cofibration and it classifies partial maps with cofibrant domain.*

In fact,  $\eta: \text{Id} \Rightarrow +$  is cartesian:

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \longrightarrow & 1 \\
 \eta_A \downarrow \ulcorner & & \downarrow \eta_B \ulcorner & & \downarrow \text{t} \\
 A^+ & \xrightarrow{f^+} & B^+ & \longrightarrow & \text{Cof}
 \end{array}$$

The right square & the outer rectangle are cartesian  $\Rightarrow$  The left square is cartesian.

## Proposition

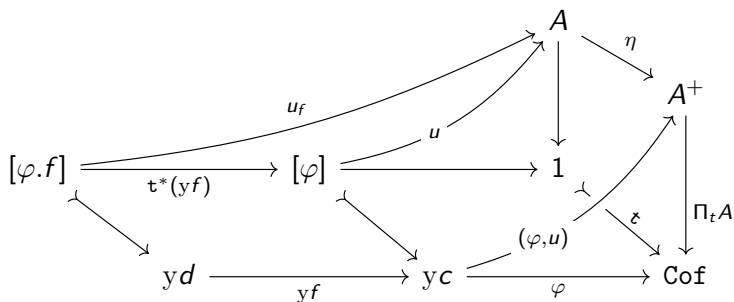
$c \Vdash [(\varphi, u) : A^+](\gamma) \Leftrightarrow$

$c \Vdash [\varphi : \text{Cof}](\gamma)$  and for all  $f: d \rightarrow c$ , if  $d \Vdash [x : \varphi.f](\gamma.f)$  then  $d \Vdash [u_f(x) : A](\gamma.f)$ ,  
where  $u_f(x).g = u_{fg}(x)$ , for all  $g: d' \rightarrow d$ .

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The above gets simplified when  $\Gamma = 1$ .

$$c \Vdash [(\varphi, u) : A^+] \quad \Leftrightarrow \quad \begin{array}{ccccc} 1 & \longleftarrow & [\varphi] & \xrightarrow{u} & A \\ \downarrow \mathfrak{t} & & \downarrow m & & \downarrow \eta_A \\ \mathbf{Cof} & \xleftarrow{\varphi} & y\mathbf{c} & \xrightarrow{(\varphi, u)} & A^+ \end{array}$$



# Monad structure from dominance

Proposition (Awodey,2018)

$+: \mathcal{E} \rightarrow \mathcal{E}$  is a (fibred) monad.

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First, we give a category-theoretic proof.

1st Proof.

$\eta_A, \eta_{A^+}$ : cofibrations  $\Rightarrow \eta_{A^+} \circ \eta_A$ : cofibration by dominance.

$\eta_A$ : cofibrant partial map classifier  $\Rightarrow$  there is a unique morphism  $\mu_A$  classifying the partial map  $(\eta_{A^+} \circ \eta_A, \text{id}_A)$ .

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \eta_A \downarrow & \lrcorner & \downarrow \eta \\ A^+ & & \\ \eta_{A^+} \downarrow & & \\ A^{++} & \xrightarrow{\quad \mu \quad} & A^+ \end{array}$$

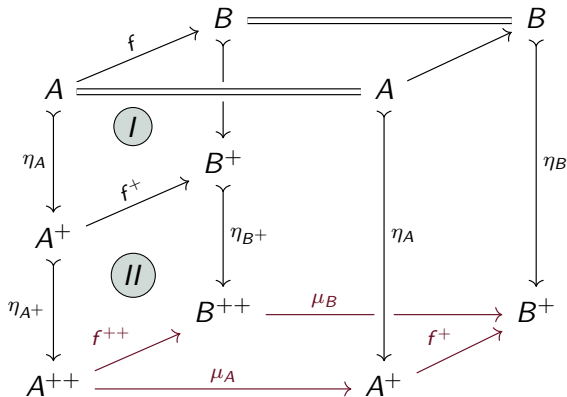
# Monad structure from dominance

(1st Proof cont'd.)

$\mu_A$  thus obtained is natural in  $A$ :

By classifying property of  $\eta_B$  the bottom square commutes since

- (i) all vertical squares are pullbacks (I and II) because  $\eta$  is cartesian),
- (ii) the top square commutes,
- (iii)  $\eta_{A^+} \circ \eta_A$ : cofibration by dominance.



## Monad structure from dominance

(1st Proof cont'd.)

To see that  $\mu \circ \eta_{A^+} = \text{id}_{A^+}$ , observe that the following is a pullback by an easy diagram chase using the previous diagram and the fact that  $\eta$  is always monic.

$$\begin{array}{ccccc} A & \xlongequal{\quad \Gamma \quad} & A & & \\ \eta_A \downarrow & & \downarrow \eta_A & & \\ A^+ & \xrightarrow{\eta_{A^+}} & A^{++} & \xrightarrow{\mu} & A^+ \end{array}$$

By the uniqueness of the classifying map of  $(\eta_A, \text{id}_A)$ , we have  $\mu_A \circ \eta_{A^+} = \text{id}_{A^+}$ . By naturality of  $\eta$ ,

$$\eta_{A^+} \circ \eta_A = (\eta_A)^+ \circ \eta_A$$

The same argument above shows

$$\mu_A \circ \eta_{A^+} = \text{id}_{A^+} .$$

Proposition ((Awodey, 2018))

$+: \mathcal{E} \rightarrow \mathcal{E}$  is a (fibred) monad.

Now, we give a proof using Kripke–Joyal semantics.

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**2nd Proof.** Write  $A^{++} = (A^+)^+$ .

$c \Vdash (\varphi, u) : A^{++}$

$\Leftrightarrow u = (\psi, u')$ ,  $c \Vdash [\varphi : \text{Cof}]$ , and for every  $f : c' \rightarrow c$ , if  $c' \Vdash [x : \varphi.f]$  then

$c' \Vdash [\psi_f(x) : \text{Cof}]$ , and for every  $g : d \rightarrow c'$ , if  $d \Vdash [y : \psi.g]$  then  $d \Vdash [u'_g(y) : A]$  and  $u'$

is uniform.

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Now, set  $f = \text{id}_c$ .

The statement above (after  $\Leftrightarrow$ ) becomes  $u = (\psi, u')$  and  $c \Vdash \varphi : \mathbf{Cof}$ ,

$c \Vdash \psi : [\varphi] \rightarrow \mathbf{Cof}$ ,  $c \Vdash u' : \sum_{x : [\varphi]} [\psi(x)] \rightarrow A$

The latter implies

$$c \Vdash \text{dom}(\varphi, \psi) : \mathbf{Cof} \text{ and } c \Vdash u' : \text{dom}(\varphi, \psi) \rightarrow A.$$



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Hence

$$c \Vdash (\text{dom}(\varphi, \psi), u') : A^+.$$

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Uniformity of  $u'$  implies  $\mathcal{E} \Vdash \mu : A^{++} \rightarrow A^+$ .

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Uniformity of  $u'$  implies  $\mathcal{E} \Vdash \mu : A^{++} \rightarrow A^+$ .

By Yoneda, we get  $\mu : A^{++} \rightarrow A^+$ .

Also,  $\mu \circ \eta_{A^+} = \text{id} = \mu \circ +(\eta_A)$  because  $\text{dom}(\varphi, \mathfrak{t}) = \varphi$  and  $\text{dom}(\mathfrak{t}, \psi) = \psi$ .

$\mu \circ \mu_{A^+} = \mu \circ +(\mu_A)$  because  $\text{dom}(\text{dom}(\varphi, \psi), \theta) = \text{dom}(\varphi, \text{dom}(\psi, \theta))$ .

For any type  $A$  define

$$\mathsf{TFib}(A) := \prod_{\varphi:\mathsf{Cof}} \prod_{u:[\varphi]\rightarrow A} \sum_{a:A} u =_{\varphi} a,$$

where the type  $u =_{\varphi} a$  (written  $(\varphi, u) \nearrow a$  in Orton and Pitts 2018) is defined

$$(u =_{\varphi} a) := \prod_{p:[\varphi]} \mathsf{Eq}_A(up, a).$$

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### Proposition

*The map  $p_A: \Gamma.A \rightarrow \Gamma$  is a uniform trivial fibration  $\Leftrightarrow$  there is a term  $\Gamma \vdash \alpha : \text{TFib}(A)$ .*

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Recall that  $p_A$  being a **a uniform trivial fibration** means that for every cofibration  $C \twoheadrightarrow Z$  and commutative square there is a diagonal filler  $j_C(z, a): Z \rightarrow \Gamma.A$  making both triangles commute,

$$\begin{array}{ccc} C & \xrightarrow{a} & \Gamma.A \\ \downarrow & \nearrow^{j_C(z,a)} & \downarrow p_A \\ Z & \xrightarrow{z} & \Gamma \end{array}$$

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$$\begin{array}{ccccc}
 C' & \xrightarrow{f'} & C & \xrightarrow{a} & \Gamma.A \\
 \downarrow \Upsilon & \lrcorner & \downarrow \Upsilon & \nearrow j_C(z, a) & \downarrow p_A \\
 Z' & \xrightarrow{f} & Z & \xrightarrow{z} & \Gamma
 \end{array}$$

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$$j_{C'}(zf, af') = j_C(z, a) \circ f.$$

$$\begin{array}{ccccc}
 C' & \xrightarrow{f'} & C & \xrightarrow{a} & \Gamma.A \\
 \downarrow \Gamma & & \downarrow & \nearrow j_{C'}(zf, af') & \downarrow p_A \\
 Z' & \xrightarrow{f} & Z & \xrightarrow{z} & \Gamma \\
 & & & \nearrow j_C(z, a) & \\
 & & & & \Gamma
 \end{array}$$

## Lemma

For  $\Gamma \vdash A \text{ Type}$ ,  $\gamma: yc \rightarrow \Gamma$  such that

$$c \Vdash a : A(\gamma)$$

$$c \Vdash \varphi : \text{Cof}(\gamma)$$

$$c \Vdash u : ([\varphi] \rightarrow A)(\gamma).$$

then we also have

$$c \Vdash e : (u =_{\varphi} a)(\gamma) \quad \Leftrightarrow \quad \begin{array}{ccc} [\varphi] & \xrightarrow{u} & \Gamma.A \\ \downarrow & \nearrow \langle \gamma, a \rangle & \downarrow p_A \\ yc & \xrightarrow{\gamma} & \Gamma \end{array} \quad \text{commutes,}$$

where

$$(u =_{\varphi} a) := \prod_{x:[\varphi]} \text{Eq}_A(ux, a).$$

Proof of Lemma.

$$\begin{array}{ccc}
 [\varphi] & \xrightarrow{u} & \Gamma.A \\
 \downarrow & \nearrow \langle \gamma, a \rangle & \downarrow p_A \\
 y\mathbf{c} & \xrightarrow{\gamma} & \Gamma
 \end{array}$$

$c \Vdash a : A(\gamma) \Leftrightarrow$  the lower triangle commutes.

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$c \Vdash a : A(\gamma) \Leftrightarrow$  the lower triangle commutes.

$c \Vdash \varphi : \mathbf{Cof}(\gamma)$  and  $c \Vdash (u : [\varphi] \rightarrow A)(\gamma) \Leftrightarrow$  the outer square commutes.

Proof of Lemma.

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 [\varphi] & \xrightarrow{u} & \Gamma.A \\
 \downarrow & \nearrow \langle \gamma, a \rangle & \downarrow p_A \\
 yc & \xrightarrow{\gamma} & \Gamma
 \end{array}$$

$c \Vdash a : A(\gamma) \Leftrightarrow$  the lower triangle commutes.

$c \Vdash \varphi : \mathbf{Cof}(\gamma)$  and  $c \Vdash (u : [\varphi] \rightarrow A)(\gamma) \Leftrightarrow$  the outer square commutes.

$c \Vdash e : u =_{\varphi} a(\gamma)$

$\Leftrightarrow c \Vdash e : \prod_{x: [\varphi]} \mathbf{Eq}_A(ux, a)(\gamma)$

$\Leftrightarrow$  for all  $f : d \rightarrow c$  in  $\mathcal{C}$ ,  $d \Vdash x : [\varphi](\gamma.f)$  returns  $d \Vdash e_f(x) : \mathbf{Eq}_A(ux, a)(\gamma.f)$

$\Leftrightarrow$  the top triangle commutes. QED.

Proof of Theorem.

Suppose  $\Gamma \vdash \alpha : \text{TFib}(A)$ .

Thus for all  $\gamma : y c \rightarrow \Gamma$ , we have  $c \Vdash \alpha_\gamma : \text{TFib}(A)(\gamma)$ , coherently in  $\gamma$ .



Proof of Theorem.

Suppose  $\Gamma \vdash \alpha : \text{TFib}(A)$ .

Thus for all  $\gamma : y_c \rightarrow \Gamma$ , we have  $c \Vdash \alpha_\gamma : \text{TFib}(A)(\gamma)$ , coherently in  $\gamma$ .

Note that

$$\begin{aligned} \text{TFib}(A) &= \prod_{\varphi : \text{Cof}} \prod_{u : [\varphi] \rightarrow A} \sum_{a : A} \prod_{x : [\varphi]} \text{Eq}_A(ux, a) \\ &= \prod_{(\varphi, u) : A^+} \sum_{a : A} u =_\varphi a \end{aligned}$$

We thus obtain

$$c \Vdash \alpha_\gamma : \prod_{(\varphi, u) : A^+} \sum_{a : A} (u =_\varphi a)(\gamma).$$

Proof of Theorem (cont'd).

By Kripke–Joyal semantics of  $\prod$  and  $\sum$ , we have for every  $f: d \rightarrow c$  in  $\mathcal{C}$ , if

$$d \Vdash (\varphi, u) : A^+(\gamma.f) \quad (1)$$

then

$$d \Vdash \alpha_{\gamma.f}(\varphi, u)^0 : A(\gamma.f) \quad (2)$$

and

$$d \Vdash \alpha_{\gamma.f}(\varphi, u)^1 : (u =_{\varphi} \alpha_{\gamma.f}(\varphi, u)^0)(\gamma.f) \quad (3)$$

and, for any  $g: d' \rightarrow d$ ,

$$\alpha_{\gamma.f}(\varphi, u) \cdot g = \alpha_{(\gamma.fg)}(\varphi[g], u[g]). \quad (4)$$

Unfolding the condition (1) yields the following commutative diagram.

$$\begin{array}{ccc} [\varphi.f] & \xrightarrow{\langle \gamma.f, u_f \rangle} & \Gamma.A \\ \downarrow & & \downarrow p_A \\ yd & \xrightarrow{\gamma.f} & \Gamma \end{array}$$

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 [\varphi.f] & \xrightarrow{\langle \gamma.f, u_f \rangle} & \Gamma.A \\
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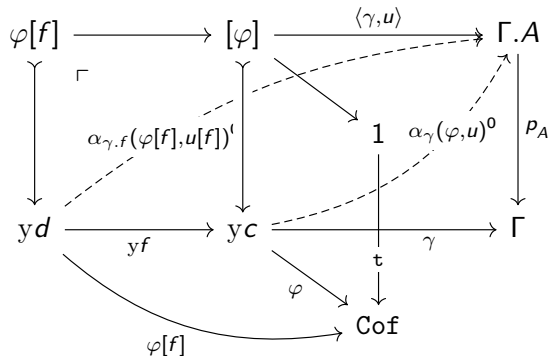
Lemma applied to (2) and (3) yields the following commuting diagram.

$$\begin{array}{ccc}
 \varphi.f & \xrightarrow{\langle \gamma.f, u_f \rangle} & \Gamma.A \\
 \downarrow & \nearrow \alpha_{\gamma.f}(\varphi, u)^0 & \downarrow p_A \\
 yd & \xrightarrow{\gamma.f} & \Gamma
 \end{array}$$

Thus forcing  $\text{TFib}(A)$  produces diagonal fillers

$$j_\varphi(\gamma, u) \triangleq \alpha_{\gamma, f}(\varphi, u)^0$$

for each lifting problem as in the right hand square below:



## Proof of Theorem (cont'd) – converse argument

If  $p_A: \Gamma.A \rightarrow \Gamma$  is a uniform trivial fibration then in particular for every *basic* cofibration  $[\varphi] \twoheadrightarrow yC$  and square as on the right below, there is a diagonal filler  $j_\varphi(\gamma, u)$  as indicated.

$$\begin{array}{ccccc}
 [\varphi.f] & \longrightarrow & [\varphi] & \xrightarrow{u} & \Gamma.A \\
 \downarrow & & \downarrow & \nearrow j_\varphi(\gamma, u) & \downarrow p_A \\
 yC' & \xrightarrow{yf} & yC & \xrightarrow{\gamma} & \Gamma
 \end{array}$$

## Proof of Theorem (cont'd) – converse argument

If  $p_A: \Gamma.A \rightarrow \Gamma$  is a uniform trivial fibration then in particular for every *basic* cofibration  $[\varphi] \twoheadrightarrow y\mathcal{C}$  and square as on the right below, there is a diagonal filler  $j_\varphi(\gamma, u)$  as indicated.

$$\begin{array}{ccccc}
 [\varphi.f] & \longrightarrow & [\varphi] & \xrightarrow{u} & \Gamma.A \\
 \downarrow & & \downarrow & \nearrow j_{\varphi.f}(\gamma.f, u.f) & \downarrow p_A \\
 y\mathcal{C}' & \xrightarrow{yf} & y\mathcal{C} & \xrightarrow{\gamma} & \Gamma
 \end{array}$$

By the lemma, this corresponds to an element  $\alpha_\gamma: y\mathcal{C} \rightarrow \text{TFib}(A)$  over  $\gamma: y\mathcal{C} \rightarrow \Gamma$ ,

$$\begin{array}{ccc}
 & \Gamma.\text{TFib}(A) & \\
 \alpha_\gamma \nearrow & \downarrow p_{\text{TFib}(A)} & \\
 y\mathcal{C} & \xrightarrow{\gamma} & \Gamma
 \end{array}$$

Proof of Theorem (cont'd) – converse argument

The uniformity condition says exactly that for all  $f : c' \rightarrow c$ , the elements  $\alpha_\gamma$  cohere,

$$\alpha_{(\gamma.yf)} = \alpha_\gamma \circ f .$$



### Proof of Theorem (cont'd) – converse argument

The uniformity condition says exactly that for all  $f : c' \rightarrow c$ , the elements  $\alpha_\gamma$  cohere,

$$\alpha_{(\gamma.yf)} = \alpha_\gamma \circ f.$$

By Yoneda for the slice category  $\mathcal{E}/\Gamma$  that there is a term  $\Gamma \vdash \alpha : \text{TFib}(A)$ . QED.

## Further Applications: A fibrant universe of fibrant types

As in (Orton and Pitts, 2018), we assume a tiny interval  $I$  in  $\mathcal{E}$  equipped with the following structures:

- ▶ Terms  $0, 1 : I$ ,
- ▶ *connections*  $\_ \sqcap \_, \_ \sqcup \_ : I \times I \rightarrow I$ ,

satisfying certain axioms.

## Further Applications: A fibrant universe of fibrant types

Using the amazing right adjoint  $(-)^I \dashv \sqrt[']{-}$  and the forcing of the type

$$\text{Fill}_e(A) = \prod_{\varphi:\text{Cof}} \prod_{u:[\varphi] \rightarrow \prod_{i:I} A_i} \prod_{a_e:A_e} (u e =^\varphi a_e) \rightarrow \sum_{a:\prod_{i:I} A_i} (a e =_{A_e} a_e) \times (u =^\varphi a),$$

we show that that the universe  $\text{Fib}$  of fibrant types is itself fibrant.






$$\begin{array}{ccccc}
 \text{Fib}(A) & \longrightarrow & \text{Fib} & \longrightarrow & \sqrt[']{\text{Type}}_{\bullet} \\
 \downarrow \text{dashed} & & \downarrow \Gamma & & \downarrow \\
 \Gamma & \xrightarrow{A} & \text{Type} & \xrightarrow{\eta} & \sqrt[']{\text{Type}}^I \xrightarrow{\sqrt[']{\text{Fill}}} \sqrt[']{\text{Type}}
 \end{array}$$

## Next ...

Further use of Kripke–Joyal semantics for dependent type theory in

- ▶ Showing Frobenius property of fibrations.
- ▶ Showing equivalence extension property.
- ▶ Extending the semantics to sheaf toposes
- ▶ Studying equivariant fibrations by means of Kripke–Joyal semantics in topos of group actions.

# References I

-  Steve Awodey. “A cubical model of homotopy type theory”. In: *Ann. Pure Appl. Logic* 169.12 (2018).
-  Steve Awodey. “Natural models of homotopy type theory”. In: *Math. Structures Comput. Sci.* 28.2 (2018).
-  Cyril Cohen et al. “Cubical type theory: a constructive interpretation of the univalence axiom”. In: *21st International Conference on Types for Proofs and Programs*. Vol. 69. 2018.
-  Nicola Gambino and Christian Sattler. “The Frobenius condition, right properness, and uniform fibrations”. In: *Journal of Pure and Applied Algebra* 221.12 (2017).
-  Ian Orton and Andrew M. Pitts. “Axioms for Modelling Cubical Type Theory in a Topos”. In: *Logical Methods in Computer Science* 14 (4 2018).

## References II



Christian Sattler. “The Equivalence Extension Property and Model Structures”. In: (2017). url: <http://arxiv.org/abs/1704.06911>.

The End

Thanks for your attention!