Kripke-Joyal Semantics for Dependent Type Theory[1]

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March 2021

[1] joint work with Steve Awodey (CMU) and Nicola Gambino (Leeds)

Outline

1. Review of classical Kripke-Joyal semantics for toposes
2. Kripke–Joyal semantics for dependent type theories
   - Forcing extensional equality types
   - Forcing dependent sum types
   - Forcing dependent product types
   - Forcing universe of small types
   - Forcing the impredicative universe Ω of propositions
   - Forcing the type of partial elements
   - Applications

Review of classical Kripke-Joyal semantics for toposes

- **The Kripke–Joyal semantics** of a topos $\mathscr{E}$ gives an interpretation to formulas written in its higher order intuitionistic internal language $HoL(\Sigma_{\mathscr{E}})$.

- **The Kripke–Joyal semantics** of a topos $\mathscr{E}$ gives an interpretation to formulas written in its higher order intuitionistic internal language $HoL(\Sigma_{\mathscr{E}})$.
- The Kripke–Joyal semantics is in fact a higher order generalization of the well-known Kripke semantic for intuitionistic propositional logic.

Definition

Let $\mathscr{E}$ be an elementary topos. Given a formula $\varphi(x)$ with a free variable x of sort A in $HoL(\Sigma_{\mathscr{E}})$, and a generalized element $\alpha \colon U \to A$ in $\mathscr{E}$, we define

$$U \Vdash \varphi(\alpha) \quad \Leftrightarrow \quad \alpha \text{ factors through the subobject } [\varphi] \rightarrowtail A.$$

$$\begin{tikzcd}
 & {[\varphi]} \arrow[r, "!"] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\texttt{true}"] \\
U \arrow[ur, dashed] \arrow[r, "\alpha"'] & X \arrow[r, "\varphi"'] & \Omega
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- Call U the **stage** of forcing.
- Write $\mathscr{E} \Vdash \varphi$ if at every stage U and for every generalized element α, we have $U \Vdash \varphi(\alpha)$.

One can then show:

- $U \Vdash \top(\alpha)$.
- $U \Vdash \bot(\alpha)$ iff U is the initial object of $\mathscr{E}$.
- $U \Vdash (x = x')(\langle \alpha, \alpha' \rangle)$ iff $\alpha \colon U \to X$ and $\alpha' \colon U \to X$ are the same maps in $\mathscr{E}$.
- $U \Vdash (\varphi \wedge \psi)(\alpha)$ iff $U \Vdash \varphi(\alpha)$ and $U \Vdash \psi(\alpha)$.
- $U \Vdash (\varphi \vee \psi)(\alpha)$ iff there are jointly epimorphic arrows $p \colon V \to U$ and $q \colon W \to U$ such that $V \Vdash \varphi(\alpha \circ p)$ and $W \Vdash \varphi(\alpha \circ q)$.
- $U \Vdash (\varphi \Rightarrow \psi)(\alpha)$ iff for any arrow $f \colon V \to U$ such that $V \Vdash \varphi(\alpha \circ f)$ then $V \Vdash \psi(\alpha \circ f)$.
- $c \Vdash \neg\varphi(\alpha)$ iff for all maps $f \colon V \to U$ in $\mathscr{E}$, $V \nVdash \varphi(\alpha.f)$.
- $\vdots$

Classical Kripke-Joyal semantics for presheaf toposes (review) (III)

- Henceforth, $\mathscr{E} = \mathcal{P}\mathsf{Shv}(\mathcal{C}) = \mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$.

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- Henceforth, $\mathscr{E} = \mathcal{P}\mathsf{Shv}(\mathcal{C}) = \mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$.
- In the presheaf toposes, every presheaf is a colimit of representables.
- So it is enough to consider forcing statements $U \Vdash \varphi(\alpha)$ for representables $U = \mathrm{y}c$.

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- So it is enough to consider forcing statements $U \Vdash \varphi(\alpha)$ for representables $U = \mathrm{y}c$.

$$c \Vdash (\varphi \vee \psi)(\alpha) \;\Leftrightarrow\; c \Vdash \varphi(\alpha) \text{ or } c \Vdash \psi(\alpha)$$

Recall $\mathrm{y}c$ is projective & indecomposable.

$$\begin{tikzcd}
 & {[\varphi] + [\psi]} \arrow[d, two heads] \\
 & {[\varphi] \cup [\psi]} \arrow[d, tail] \\
\mathrm{y}c \arrow[ruu, dashed] \arrow[ru, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

Limitations of classical Kripke–Joyal semantics

- Bounded quantification. We shall overcome this by generalizing Kripke–Joyal semantics to dependent type theory with universes.
- Equality of terms is extensional and not “up to homotopy’. We will also generalize to homotopy type theory.

Kripke–Joyal semantics for dependent type theory

- Provability versus proof relevance: the classical Kripke–Joyal semantics is only concerned with the provability of a proposition; it is proof irrelevant. Kripke–Joyal semantics for dependent type theory includes the terms (proofs) in the forcing statements.

Kripke–Joyal semantics for dependent type theory

- Provability versus proof relevance: the classical Kripke–Joyal semantics is only concerned with the provability of a proposition; it is proof irrelevant. Kripke–Joyal semantics for dependent type theory includes the terms (proofs) in the forcing statements.
- We want a sound, formal and *(quasi-) mechanical* process to relate internal developments (Cohen et al., 2018), (Orton and Pitts, 2018), etc. with the diagrammatic developments (Gambino and Sattler, 2017), (Sattler, 2017), (Awodey, 2018), etc. found in the models of HoTT literature.

Kripke–Joyal semantics for dependent type theories

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- We work in ZFC extended with two inaccessible cardinals $\kappa < \lambda$.

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- We fix a λ-small category $\mathcal{C}$. We define the Grothendieck topos of presheaves

$$\mathscr{E} = [\mathcal{C}^{\mathrm{op}}, \mathcal{S}\mathsf{et}]$$

We write

$$\mathrm{y} : \mathcal{C} \to \mathscr{E}$$

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- As usual, we get classifiers $\pi \colon \mathsf{Type}^{\bullet} \to \mathsf{Type}$ and $\pi_{\mathcal{V}} \colon \mathcal{V}^{\bullet} \to \mathcal{V}$ for λ-small and κ-small families, resp.

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The internal dependent type theory of presheaf toposes (II)

Following (Awodey, 2018), we get a CwF structure on $\mathscr{E}$ from the universe $\pi : \mathsf{Type}^{\bullet} \to \mathsf{Type}$:

Steve Awodey. "Natural models of homotopy type theory". In: *Math. Structures Comput. Sci.* 28.2 (2018)

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A **term** $a\colon A$ in context Γ is a map $a\colon \Gamma \to \mathsf{Type}^\bullet$ with $\pi \circ a = A$.

$$\begin{tikzcd} & \mathsf{Type}^\bullet \arrow[d, "\pi"] \\ \Gamma \arrow[ur, dotted, "a"] \arrow[r, "A"'] & \mathsf{Type} \end{tikzcd}$$

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The **context extension** is given by the pullback along π.

$$\begin{tikzcd} \Gamma.A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathsf{Type}^\bullet \arrow[d, "\pi"] \\ \Gamma \arrow[r, "A"'] & \mathsf{Type} \end{tikzcd}$$

$$\frac{A \in \mathrm{Type}(\Gamma) \qquad \gamma\colon \Delta \to \Gamma}{A(\gamma) \triangleq A \circ \gamma \in \mathrm{Type}(\Delta)}\ \text{(substitution-types)}$$

$$\frac{a \in \mathrm{Term}(\Gamma, A) \qquad \gamma\colon \Delta \to \Gamma}{a(\gamma) \triangleq a \circ \gamma \in \mathrm{Term}(\Delta, A(\gamma))}\ \text{(substitution-terms)}$$

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When $\Delta = \mathrm{y}c$:

by the Yoneda lemma, $\gamma \in \Gamma(c)$, and $A(\gamma) \in \mathrm{Type}(c)$.

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by the Yoneda lemma, $\gamma \in \Gamma(c)$, and $A(\gamma) \in \mathrm{Type}(c)$.

For a morphism $f\colon d \to c$ in $\mathcal{C}$, we write $\gamma.f$ for the composite $\gamma \circ \mathrm{y}f$.

Also, for $a \in \mathrm{Term}(\mathrm{y}c, A(\gamma))$ and we have $a(\gamma.f) = a(\gamma).f \in \mathrm{Term}(d, A(\gamma.f))$.

The internal dependent type theory of presheaf toposes (IV)

$\mathcal{T}(\mathscr{E})$ = the internal dependent type theory of presheaf toposes.

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$\mathcal{T}(\mathscr{E})$ has

- the basic types $\emptyset$, $\mathbf{1}$, $\mathbf{2}$, N.
- also the following forms of type:

$$A \times B\,, \quad A + B\,, \quad A \to B$$
$$\mathrm{Eq}_A(a, b)\,, \quad \sum_{x:A} B(x)\,, \quad \prod_{x:A} B(x)$$

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- The type of extensional equality is given by the diagonal map $\delta \colon A \rightarrowtail A \times A$.
- These types satisfy the usual induction and computation rules, e.g. in HoTT-Book.
- There is an evident tautological interpretation of $\mathcal{T}(\mathscr{E})$ into $\mathscr{E}$, using the CwF structure of $\mathscr{E}$.

$\mathcal{T}(\mathscr{E})$ additionally has the types

- $\mathcal{V}$: the universe of small types

$$\frac{a \in \mathrm{Term}(\Gamma, \mathrm{v})}{\mathrm{El}(a) \in \mathrm{Term}(\Gamma, \mathrm{Type})}$$

$$\begin{tikzcd}
\Gamma.\mathrm{El}(a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{V}^\bullet \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Type}^\bullet \arrow[d] \\
\Gamma \arrow[r, "a"'] & \mathcal{V} \arrow[r, tail, "\mathrm{El}"'] & \mathrm{Type}
\end{tikzcd}$$

The internal dependent type theory of presheaf toposes (IV)

$\mathcal{T}(\mathscr{E})$ additionally has the types

- $\mathcal{V}$: the universe of small types
- Ω: the impredicative universe of propositions.

$$\frac{\varphi \in \mathsf{Term}(\Gamma, \omega)}{\iota(\varphi) \in \mathsf{Term}(\Gamma, \mathsf{Type})}$$

$$\begin{tikzcd}
{[\varphi]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[r, "\widetilde{*}"] \arrow[d, tail, "\mathtt{true}"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{Type}^{\bullet} \arrow[d, "\pi"] \\
\Gamma \arrow[r, "\varphi"'] & \Omega \arrow[r, "\iota"'] & \mathsf{Type}
\end{tikzcd}$$

The internal dependent type theory of presheaf toposes (IV)

$\mathcal{T}(\mathscr{E})$ additionally has the types

- $\mathcal{V}$: the universe of small types
- Ω: the impredicative universe of propositions.
- Note that

$$\mathsf{El}(\iota(\varphi)) \cong [\varphi]$$

As in (Orton and Pitts, 2018), we consider a modality $\mathtt{cof} : \Omega \to \Omega$ satisfying:

(i) $\mathtt{cof} \circ \mathtt{true} = \mathtt{true}$,

(ii) $\mathtt{cof} \circ \mathtt{false} = \mathtt{true}$,

(iii) $\forall(\varphi, \psi : \Omega).\ \mathtt{cof}\, \varphi \Rightarrow (\varphi \Rightarrow \mathtt{cof}\, \psi) \Rightarrow \mathtt{cof}(\varphi \wedge \psi)$.

The internal dependent type theory of presheaf toposes (V)

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The last axiom is called the **principle of dominance**.

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Obtain $m_{\mathtt{Cof}} : \mathtt{Cof} \rightarrowtail \Omega$ as the comprehension subtype.

$$\mathtt{Cof} \triangleq \{\varphi \in \Omega \mid \mathtt{cof}\, \varphi\}$$

$$\begin{tikzcd}
1 \arrow[r, tail, "\mathtt{t}"] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner", very near start] & \mathtt{Cof} \arrow[r] \arrow[d, tail, "m_{\mathtt{Cof}}"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathtt{true}"] \\
1 \arrow[r, tail, "\mathtt{true}"'] & \Omega \arrow[r, "\mathtt{cof}"'] & \Omega
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Dependent Kripke–Joyal semantics

Definition (Dependent Kripke–Joyal semantics– forcing terms)

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For a context Γ, a type $\Gamma \vdash A \ \mathtt{Type}$, an object c of $\mathcal{C}$,

$$\mathrm{y}c \qquad \begin{tikzcd} \Gamma.A \arrow[r] \arrow[d] & \mathsf{Type}^{\bullet} \arrow[d] \\ \Gamma \arrow[r, "A"'] & \mathsf{Type} \end{tikzcd}$$

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For a context Γ, a type $\Gamma \vdash A$ $\mathtt{Type}$, an object c of $\mathcal{C}$, and a morphism $\gamma \colon \mathrm{y}c \to \Gamma$,

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For a context Γ, a type $\Gamma \vdash A\ \mathtt{Type}$, an object c of $\mathcal{C}$, and a morphism $\gamma\colon \mathrm{y}c \to \Gamma$,

$$c \Vdash a : A(\gamma) \;\Leftrightarrow\; \text{there is a lift } \langle\gamma, a\rangle \text{ of } \gamma \text{ against } p_A : \Gamma.A \to \Gamma.$$

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$c \Vdash a : A(\gamma) \;\Leftrightarrow\;$ there is a lift $\langle \gamma, a\rangle$ of γ against $p_A \colon \Gamma.A \to \Gamma$.

$$\begin{tikzcd} & \Gamma.A \arrow[r] \arrow[d] & \mathrm{Type}^{\bullet} \arrow[d] \\ \mathrm{y}c \arrow[ur, dashed] \arrow[urr, dashed, "{\langle \gamma, a\rangle}"'] \arrow[r, "\gamma"'] & \Gamma \arrow[r, "A"'] & \mathrm{Type} \end{tikzcd}$$

Proposition

$\Gamma \vdash a : A \;\Leftrightarrow\;$ *There is a family* $(a_\gamma \mid c : \textit{an object of } \mathcal{C}, \gamma \colon \mathrm{y}c \to \Gamma)$ *satisfying*

$$c \Vdash a_\gamma : A(\gamma)$$

and for every morphism $f \colon d \to c$ *of* $\mathcal{C}$,

$$a_\gamma.f = a_{\gamma.f}$$

Forcing extensional equality types

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Given a context Γ*, a type* $\Gamma \vdash A\ \mathtt{Type}$*,*

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Proposition

Given a context Γ*, a type* $\Gamma \vdash A \ \mathit{Type}$*, an object* c *of* $\mathcal{C}$*, a morphism* $\gamma \colon \mathrm{y}c \to \Gamma$*,* $c \Vdash (a, a') : (A \times A)(\gamma)$

$$
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 & \Gamma.A \arrow[d, tail, "\delta"] \\
 & A \times_\Gamma A \arrow[d] \\
\mathrm{y}c \arrow[ur, "{\langle \gamma, (a, a') \rangle}"] \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}
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Proposition

Given a context Γ*, a type* $\Gamma \vdash A\ \mathtt{Type}$*, an object* c *of* $\mathcal{C}$*, a morphism* $\gamma\colon \mathrm{y}c \to \Gamma$*,* $c \Vdash (a, a') : (A \times A)(\gamma)$ *we have*

$$c \Vdash e : \mathrm{Eq}_A(a, a')(\gamma) \ \Leftrightarrow$$

$$a, a' \text{ are equal as morphisms in } \mathscr{E} \ \Leftrightarrow$$

$$a, a' \text{ are equal elements of } A(c)\ .$$

The Type Eq_A is interpreted by the diagonal morphism $\delta\colon A \rightarrowtail A \times_\Gamma A$ over Γ.

$$\begin{tikzcd}
 & \Gamma.A \arrow[d, tail, "\delta"] \\
 & A \times_\Gamma A \arrow[d] \\
\mathrm{y}c \arrow[uur, dashed, "{\langle\langle\gamma,(a,a')\rangle,e\rangle}"] \arrow[ur, "{\langle\gamma,(a,a')\rangle}"] \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}$$

Forcing dependent sum types

Proposition

Given a context Γ,

Γ

Forcing dependent sum types

Proposition

Given a context Γ*, a type* $\Gamma \vdash A\ \mathtt{Type}$*,*

$$\begin{tikzcd} \Gamma.A \arrow[dr, "p_A"] & \\ & \Gamma \end{tikzcd}$$

Forcing dependent sum types

Proposition

Given a context Γ, *a type* $\Gamma \vdash A \; \mathtt{Type}$, *a type* $\Gamma, x : A \vdash B \; \mathtt{Type}$,

$$
\begin{tikzcd}
\Gamma.A.B \arrow[d, "p_B"'] \arrow[r, equal] & \Gamma.\sum_{a:A} B(a) \arrow[d, "p_A p_B"] \\
\Gamma.A \arrow[r, "p_A"] & \Gamma
\end{tikzcd}
$$

Forcing dependent sum types

Proposition

Given a context Γ, a type $\Gamma \vdash A$ $\mathtt{Type}$, a type $\Gamma, x : A \vdash B$ $\mathtt{Type}$, an object c of $\mathcal{C}$,

$$
\begin{tikzcd}
 & \Gamma.A.B \arrow[dr, equal] \arrow[d, "p_B"] & \\
 & \Gamma.A \arrow[ddr, "p_A"] & \Gamma.\sum_{a:A} B(a) \arrow[dd, "p_A p_B"] \\
 & & \\
\mathrm{y}c & & \Gamma
\end{tikzcd}
$$

Forcing dependent sum types

Proposition

Given a context Γ, a type $\Gamma \vdash A$ $\mathtt{Type}$, a type $\Gamma, x : A \vdash B$ $\mathtt{Type}$, an object c of $\mathcal{C}$, and a morphism $\gamma \colon \mathrm{y}c \to \Gamma$,

$$\begin{tikzcd}
& \Gamma.A.B \arrow[dd, "p_B"'] \arrow[dr, equal] & \\
& & \Gamma.\sum_{a:A} B(a) \arrow[dd, "p_A p_B"] \\
& \Gamma.A \arrow[dr, "p_A"] & \\
\mathrm{y}c \arrow[rr, "\gamma"'] & & \Gamma
\end{tikzcd}$$

Forcing dependent sum types

Proposition

Given a context Γ*, a type* $\Gamma \vdash A$ $\mathtt{Type}$*, a type* $\Gamma, x : A \vdash B$ $\mathtt{Type}$*, an object* c *of* $\mathcal{C}$*, and a morphism* $\gamma \colon \mathrm{y}c \to \Gamma$*,*

$$c \Vdash d : \left(\sum_{a:A} B(a) \right)(\gamma)$$

iff

$$d = (d_0, d_1)$$

$$c \Vdash d_0 : A(\gamma)$$

$$c \Vdash d_1 : B(\langle \gamma, d_0 \rangle) \ .$$

$$\begin{tikzcd}
 & \Gamma.A.B \arrow[d, "p_B"] \arrow[r, equal] & \Gamma.\sum_{a:A} B(a) \arrow[dd, "p_A p_B"] \\
 & \Gamma.A \arrow[dr, "p_A"] & \\
\mathrm{y}c \arrow[uur, bend left, "\langle\langle\gamma,d_0\rangle,d_1\rangle"] \arrow[ur, "\langle\gamma,d_0\rangle"] \arrow[uurr, dashed, "\langle\gamma,d\rangle"] \arrow[rr, "\gamma"'] & & \Gamma
\end{tikzcd}$$

Forcing dependent product types

Proposition

Given a context Γ,

Γ

Forcing dependent product types

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$$\begin{tikzcd} \Gamma.A \arrow[dr, "p_A"] & \\ & \Gamma \end{tikzcd}$$

Forcing dependent product types

Proposition

Given a context Γ*, a type* $\Gamma \vdash A \; \mathtt{Type}$*, a type* $\Gamma, x : A \vdash B \; \mathtt{Type}$*,*

$$\begin{tikzcd}
\Gamma.A.B \arrow[d, "\mathsf{p}_B"] & \\
\Gamma.A \arrow[dr, "\mathsf{p}_A"] & \Gamma.\Pi AB \arrow[d, "p_{\Pi_A B}"] \\
& \Gamma
\end{tikzcd}$$

Forcing dependent product types

Proposition

Given a context Γ, a type $\Gamma \vdash A$ Type, a type $\Gamma, x : A \vdash B$ Type, an object c of $\mathcal{C}$,

$$
\begin{tikzcd}
 & \Gamma.A.B \arrow[dd, "\mathsf{p}_\mathsf{B}"] & \\
 & & \Gamma.\Pi AB \arrow[dd, "p_{\Pi_A B}"] \\
 & \Gamma.A \arrow[dr, "\mathsf{p}_\mathsf{A}"] & \\
\mathrm{y}c & & \Gamma
\end{tikzcd}
$$

Forcing dependent product types

Proposition

Given a context Γ, a type $\Gamma \vdash A\ \mathtt{Type}$, a type $\Gamma, x : A \vdash B\ \mathtt{Type}$, an object c of $\mathcal{C}$, and a morphism $\gamma : \mathrm{y}c \to \Gamma$,

$$\begin{tikzcd}
 & \Gamma.A.B \arrow[d, "\mathsf{p}_B"] & \Gamma.\Pi AB \arrow[dd, "p_{\Pi_A B}"] \\
 & \Gamma.A \arrow[dr, "\mathsf{p}_A"] & \\
\mathrm{y}c \arrow[rr, "\gamma"'] & & \Gamma
\end{tikzcd}$$

Forcing dependent product types

Proposition

Given a context Γ*, a type* $\Gamma \vdash A$ $\mathtt{Type}$*, a type* $\Gamma, x : A \vdash B$ $\mathtt{Type}$*, an object* c *of* $\mathcal{C}$*, and a morphism* $\gamma : \mathrm{y}c \to \Gamma$,

$$c \Vdash b : \left(\prod_{x:A} B\right)(\gamma)$$

iff there is a function b *such for every morphism* $f : d \to c$ *in* $\mathcal{C}$*, if*

$$d \Vdash a : A(\gamma.f)$$

then

$$d \Vdash b_f(a) : B(\langle \gamma.f, a \rangle)$$

and for every $g : d' \to d$*,* $b_f(a).g = b_{f \circ g}(a.g)$*.*

$$\begin{tikzcd}
 & & \Gamma.A.B \arrow[d, "p_B"] & \\
 & & & \Gamma.\Pi AB \arrow[dd, "p_{\Pi_A B}"] \\
\mathrm{y}d \arrow[rr, "\langle \gamma.f, a \rangle"] \arrow[urr, dashed, bend left, "(\langle \gamma.f, a \rangle, b_f(a))"] \arrow[dr, "\mathrm{y}(f)"'] & & \Gamma.A \arrow[dr, "p_A"] & \\
 & \mathrm{y}c \arrow[rr, "\gamma"'] \arrow[uurr, dashed, bend left, "\langle \gamma, b \rangle"] & & \Gamma
\end{tikzcd}$$

Forcing universe $\mathcal{V}$ of small types

Proposition

For an object c of $\mathcal{C}$,

$$c \Vdash a : \mathrm{v} \;\Leftrightarrow c \Vdash \mathsf{El}(a)\mathsf{Type},$$
$$\mathsf{El}(a.f) \equiv \mathsf{El}(a).f \;\; \textit{for every } f : d \to c\,, \textit{ and}$$
$$\mathsf{El}(a) \to \mathrm{y}c \;\textit{ and } \mathsf{El}(a.f) \to \mathrm{y}d \;\; (\textit{for all } f : d \to c) \textit{ are small.}$$

Forcing universe $\mathcal{V}$ of small types

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Proposition

For an object c of $\mathcal{C}$,

$$c \Vdash [a^\bullet : \mathcal{V}^\bullet] \Leftrightarrow a^\bullet = (a, b) \textit{ such that } c \Vdash a : \mathrm{v}$$
$$\textit{and } c \Vdash b : \mathrm{El}(a)$$

$$\begin{tikzcd}
\mathrm{El}(a) \arrow[r, "\mathsf{q}_a"] \arrow[d, "p_a"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{V}^\bullet \arrow[d, "p_{\mathcal{V}}"] \\
\mathrm{y}c \arrow[u, bend left, "b"] \arrow[r, "a"'] & \mathcal{V}
\end{tikzcd}$$

Forcing Ω

Theorem

Let $\varphi\colon \Gamma \to \Omega$ and $\gamma\colon c \to \Gamma$. Then the following are equivalent:

1. *$c \Vdash \varphi(\gamma)$ in the sense of the standard Kripke-Joyal semantics,*
2. *there exists a (necessarily unique) $a\colon \mathrm{y}c \to \mathcal{V}^\bullet$ such that $c \Vdash a\colon \iota\varphi(\gamma)$.*

$$
\begin{tikzcd}
 & {[\varphi]} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[r] \arrow[d, tail, "\mathtt{true}"] & \mathsf{Type}^\bullet \arrow[d, "\pi"] \\
\mathrm{y}c \arrow[r, "\gamma"'] \arrow[ur, dotted] & \Gamma \arrow[r, "\varphi"'] & \Omega \arrow[r, "\iota"'] & \mathsf{Type}
\end{tikzcd}
$$

Cofibrations

Definition

A monomorphism $m\colon C \rightarrowtail Z$ is a **cofibration** if its classifying map $\chi_m\colon Z \to \Omega$ factors through $m_{\mathtt{cof}}\colon \mathtt{Cof} \rightarrowtail \Omega$.

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[r] \arrow[d, tail, "\mathtt{t}"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathtt{true}"] \\ Z \arrow[r, dashed] \arrow[rr, bend right, "\chi_m"'] & \mathtt{Cof} \arrow[r, tail, "m_{\mathtt{cof}}"] & \Omega \end{tikzcd}$$

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C \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[r] \arrow[d, tail, "\mathtt{t}"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathtt{true}"] \\
Z \arrow[r, dashed] \arrow[rr, bend right, "\chi_m"'] & \mathtt{Cof} \arrow[r, tail, "m_{\mathtt{cof}}"] & \Omega
\end{tikzcd}$$

Therefore, all cofibrations are the pullbacks of the generic cofibration $\mathtt{t}\colon 1 \rightarrowtail \mathtt{Cof}$.

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$$
\begin{tikzcd}
C \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[r] \arrow[d, tail, "\mathtt{t}"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathtt{true}"] \\
Z \arrow[r, dashed] \arrow[rr, bend right, "\chi_m"'] & \mathtt{Cof} \arrow[r, tail, "m_{\mathtt{cof}}"] & \Omega
\end{tikzcd}
$$

Therefore, all cofibrations are the pullbacks of the generic cofibration $\mathtt{t} \colon 1 \rightarrowtail \mathtt{Cof}$.

Proposition

$m \colon C \rightarrowtail Z$ is a cofibration $\Leftrightarrow$ $\mathscr{E} \Vdash \forall z : Z.\, \mathtt{cof}(\exists c : C.\, m(c) = z)$.

Forcing dominance (I)

Consider the following polynomials

$$1 \xrightarrowtail{\mathtt{t}} \mathtt{Cof}$$

$$1 \xrightarrowtail{\mathtt{true}} \Omega$$

$$\mathsf{Type}^{\bullet} \xrightarrow{p_{\mathsf{Type}}} \mathsf{Type}$$

associated polynomial functor

$$\mathscr{E} \xrightarrow{P_{\mathtt{t}}} \mathscr{E}$$

$$\mathscr{E} \xrightarrow{P_{\mathtt{true}}} \mathscr{E}$$

$$\mathscr{E} \xrightarrow{P_{p_{\mathsf{Type}}}} \mathscr{E}$$

where

$$P_{\mathtt{t}}(A) = \sum_{\varphi : \mathtt{Cof}} A^{[\varphi]}$$

$$P_{\mathtt{true}}(A) = \sum_{\varphi : \Omega} A^{[\varphi]}$$

$$P_{p_{\mathsf{Type}}}(A) = \sum_{a : \mathsf{Type}} A^{\mathsf{El}(a)}$$

Because the square

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, tail, "\mathtt{t}"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathtt{true}"] \\ \mathtt{Cof} \arrow[r, tail, "m_{\mathtt{cof}}"] & \Omega \end{tikzcd}$$

is cartesian, we obtain a cartesian square:

$$\begin{tikzcd} P_{\mathtt{t}}(\mathtt{Cof}) \arrow[r, tail] \arrow[d, tail, "P_{\mathtt{t}}(m_{\mathtt{cof}})"'] \arrow[dr, phantom, "\ulcorner", very near start] & P_{\mathtt{true}}(\mathtt{Cof}) \arrow[d, tail, "P_{\mathtt{true}}(m_{\mathtt{cof}})"] \\ P_{\mathtt{t}}(\Omega) \arrow[r, tail] & P_{\mathtt{true}}(\Omega) \end{tikzcd} = \begin{tikzcd} \sum_{\varphi : \mathtt{Cof}} \mathtt{Cof}^{[\varphi]} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & \sum_{\varphi : \Omega} \mathtt{Cof}^{[\varphi]} \arrow[d, tail] \\ \sum_{\varphi : \mathtt{Cof}} \Omega^{[\varphi]} \arrow[r, tail] & \sum_{\varphi : \Omega} \Omega^{[\varphi]} \end{tikzcd}$$

And because

$$\begin{tikzcd} 1 \arrow[r, "\widetilde{*}"] \arrow[d, tail, "\mathtt{true}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{Type}^{\bullet} \arrow[d, "p_{\mathsf{Type}}"] \\ \Omega \arrow[r, tail, "\iota"'] & \mathsf{Type} \end{tikzcd}$$

is cartesian, we obtain a cartesian square:

$$\begin{tikzcd} P_{\mathtt{true}}(\Omega) \arrow[r, tail] \arrow[d, tail, "P_{\mathtt{true}}(\iota)"'] \arrow[dr, phantom, "\ulcorner", very near start] & P_{p_{\mathsf{Type}}}(\Omega) \arrow[d, tail, "P_{p_{\mathsf{Type}}}(\iota)"] \\ P_{\mathtt{true}}(\mathsf{Type}) \arrow[r, tail] & P_{p_{\mathsf{Type}}}(\mathsf{Type}) \end{tikzcd} = \begin{tikzcd} \sum_{\varphi : \Omega} \Omega^{[\varphi]} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & \sum_{a : \mathsf{Type}} \Omega^{\mathsf{El}(a)} \arrow[d, tail] \\ \sum_{\varphi : \Omega} \mathsf{Type}^{[\varphi]} \arrow[r, tail] & \sum_{a : \mathsf{Type}} \mathsf{Type}^{\mathsf{El}(a)} \end{tikzcd}$$

Forcing dominance (II)

$$
\begin{tikzcd}
P_{\mathtt{t}}(\mathtt{Cof}) \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, very near start, "\ulcorner"] & P_{\mathtt{true}}(\mathtt{Cof}) \arrow[d, tail] & \\
P_{\mathtt{t}}(\Omega) \arrow[r, tail] & P_{\mathtt{true}}(\Omega) \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, very near start, "\ulcorner"] & P_{p_{\mathsf{Type}}}(\Omega) \arrow[d, tail] \\
& P_{\mathtt{true}}(\mathsf{Type}) \arrow[r, tail] & P_{p_{\mathsf{Type}}}(\mathsf{Type})
\end{tikzcd}
$$

Therefore, there is a composite map

$$\sum_{\varphi : \mathtt{Cof}} \mathtt{Cof}^{[\varphi]} = P_{\mathtt{t}}(\mathtt{Cof}) \rightarrowtail P_{\mathtt{true}}(\Omega) \rightarrowtail P_{p_{\mathsf{Type}}}(\mathsf{Type}) = \sum_{a : \mathsf{Type}} \mathsf{Type}^{\mathsf{El}(a)}$$

which takes (φ, ψ) to $(\iota\varphi, \iota\psi)$.

Proposition

$\mathscr{E} \Vdash [\mathrm{dom} : \forall(\varphi, \psi : \Omega).\ \mathtt{cof}\,\varphi \Rightarrow (\varphi \Rightarrow \mathtt{cof}\,\psi) \Rightarrow \mathtt{cof}(\varphi \wedge \psi)] \Leftrightarrow$ *there is a lift* dom *of* Σ *making the square commute.*

$$\begin{tikzcd} P_{\mathtt{t}}(\mathtt{Cof}) \arrow[r, dashed, "\mathrm{dom}"] \arrow[d, tail] & \mathtt{Cof} \arrow[d, tail, "\iota"] \\ P_{p_{\mathrm{Type}}}(\mathrm{Type}) \arrow[r, "\Sigma"'] & \mathrm{Type} \end{tikzcd}$$

- Note that $\Sigma\colon P_{p_{\mathrm{Type}}}(\mathrm{Type}) \to (\mathrm{Type})$ in above is the Natural Model (resp. CwF) interpretation of the $\sum$ type-former following (Awodey, 2018).

Forcing dominance (IV)

Proposition

For $\varphi : \texttt{Cof}$ and $\psi : [\varphi] \to \texttt{Cof}$, the following statements hold:

(i) $\mathsf{dom}(\texttt{t}, \varphi) = \varphi = \mathsf{dom}(\varphi, \texttt{t})$.

(ii) $\mathsf{dom}(\mathsf{dom}(\varphi, \psi), \theta) = \mathsf{dom}(\varphi, \mathsf{dom}(\psi, \theta))$.

(iii) $[\mathsf{dom}(\varphi, \psi)] \equiv \sum_{x : [\varphi]} [\psi(x)]$.

Proof.

For (i), note that $\iota(\texttt{t}) = \mathsf{code}(\mathbf{1})$ where $\mathbf{1}$ is the terminal type. Since $\sum_{* : \mathbf{1}} \varphi(*) = \iota\varphi$ and ι is monic, $\mathsf{dom}(\texttt{t}, \varphi) = \varphi$.

For (ii), since $\sum_{x : \iota\varphi} \texttt{t} \cong \mathsf{code}(1)$ and the "exchange rule" of the sum types.

For (iii), observe that

$$[\mathsf{dom}(\varphi, \psi)] \equiv \mathsf{El}\iota(\mathsf{dom}(\varphi, \psi)) \equiv \mathsf{El}(\Sigma(\iota\varphi, \iota\psi)) \equiv \sum_{x : [\varphi]} [\psi(x)] .$$

$\square$

Forcing dominance (V)

Proposition

Cofibrations are closed under composition.

Forcing dominance (V)

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Cofibrations are closed under composition.

Proof.

It suffices to prove that if $m_\varphi \colon [\varphi] \rightarrowtail \mathrm{y}c$ and $m_\psi \colon [\psi] \rightarrowtail [\varphi]$ are cofibrations then so is their composite.

Forcing dominance (V)

Proposition

Cofibrations are closed under composition.

Proof.

It suffices to prove that if $m_\varphi : [\varphi] \rightarrowtail \mathrm{y}c$ and $m_\psi : [\psi] \rightarrowtail [\varphi]$ are cofibrations then so is their composite.

$c \Vdash \varphi : \texttt{Cof}$ and $c \Vdash \psi : [\varphi] \to \texttt{Cof}$ imply $c \Vdash \mathrm{dom}(\varphi, \psi) : \texttt{Cof}$

Forcing dominance (V)

Proposition

Cofibrations are closed under composition.

Proof.

It suffices to prove that if $m_\varphi : [\varphi] \rightarrowtail \mathtt{y}c$ and $m_\psi : [\psi] \rightarrowtail [\varphi]$ are cofibrations then so is their composite.

$c \Vdash \varphi : \mathtt{Cof}$ and $c \Vdash \psi : [\varphi] \to \mathtt{Cof}$ imply $c \Vdash \mathrm{dom}(\varphi, \psi) : \mathtt{Cof}$

$\mathrm{dom}(\varphi, \psi)$ classifies $m_\varphi \circ m_\psi$ since (i) $[\mathrm{dom}(\varphi, \psi)] \equiv \sum_{x : [\varphi]} [\psi(x)]$, and (ii) $m_\varphi \circ m_\psi$ is the display map of the sum type $\sum_{x : [\varphi]} [\psi(x)]$.

$$\begin{tikzcd}
{[\psi]} \arrow[r] \arrow[d, tail, "m_\psi"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathtt{t}"] & \\
{[\varphi]} \arrow[r, "\psi"] \arrow[d, tail, "m_\varphi"'] \arrow[drr, phantom, "\ulcorner", very near start] & \mathtt{Cof} \arrow[r] & 1 \arrow[d, tail, "\mathtt{t}"] \\
\mathtt{y}c \arrow[rr, tail, "\varphi"'] \arrow[rr, tail, dashed, bend right, "\mathrm{dom}(\varphi,\psi)"'] & & \mathtt{Cof}
\end{tikzcd}$$

The **type of partial elements** of a type A is given by the polynomial functor

$$P_{\mathtt{true}}(A) = \sum_{\varphi : \Omega} [\varphi] \to A \,.$$

Forcing for partial elements (I)

The **type of partial elements** of a type A is given by the polynomial functor

$$P_{\texttt{true}}(A) = \sum_{\varphi : \Omega} [\varphi] \to A \,.$$

The type of **cofibrant partial elements** of a type A is given by the polynomial functor

$$A^{+} = P_{\texttt{t}}(A) = \sum_{\varphi : \texttt{Cof}} [\varphi] \to A \,.$$

There is a natural map

$$\begin{aligned} \eta\colon A &\longrightarrow A^{+} \\ a &\longmapsto (\mathtt{true}, \lambda * . a : 1 \to A) \end{aligned}$$

There is a natural map

$$\begin{aligned} \eta\colon A &\longrightarrow A^+ \\ a &\longmapsto (\mathtt{true}, \lambda * . a : 1 \to A) \end{aligned}$$

which fits into the pullback square

$$\begin{tikzcd} A \arrow[r, tail, "\eta"] \arrow[d, "!_A"'] \arrow[dr, phantom, "\ulcorner", very near start] & A^+ \arrow[d, "\mathtt{fst}"] \\ 1 \arrow[r, tail, "\mathtt{t}"'] & \mathtt{Cof} \end{tikzcd} .$$

Proposition (Awodey,2018)

The map $\eta_A\colon A \to A^+$ is a cofibration and it classifies partial maps with cofibrant domain.

Forcing for partial elements (III)

In fact, $\eta\colon \mathsf{Id} \Rightarrow + $ is cartesian:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "\eta_A"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r] \arrow[d, tail, "\eta_B"] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, "\mathtt{t}"] \\
A^{+} \arrow[r, "f^{+}"] \arrow[rr, bend right] & B^{+} \arrow[r] & \mathtt{Cof}
\end{tikzcd}
$$

The right square & the outer rectangle are cartesian $\Rightarrow$ The left square is cartesian.

Proposition

$c \Vdash [(\varphi, u) : A^{+}](\gamma) \Leftrightarrow$
$c \Vdash [\varphi : \mathtt{Cof}](\gamma)$ *and for all* $f : d \to c$, *if* $d \Vdash [x : \varphi.f](\gamma.f)$ *then* $d \Vdash [u_f(x) : A](\gamma.f)$, *where* $u_f(x).g = u_{fg}(x)$, *for all* $g : d' \to d$.

Forcing for partial elements (IV)

Proposition

$c \Vdash [(\varphi, u) : A^+](\gamma) \;\Leftrightarrow$
$c \Vdash [\varphi : \mathtt{Cof}](\gamma)$ *and for all* $f : d \to c$, *if* $d \Vdash [x : \varphi.f](\gamma.f)$ *then* $d \Vdash [u_f(x) : A](\gamma.f)$,
where $u_f(x).g = u_{fg}(x)$, *for all* $g : d' \to d$.

$$
\begin{tikzcd}
 & & & & A \arrow[dd] \arrow[dr, "\eta"] & \\
 & & & & & A^+ \arrow[dd, "\Pi_t A"] \\
{[\varphi.f]} \arrow[uurrrr, bend left, "u_f"] \arrow[rr, "\mathtt{t}^*(\mathrm{y}f)"'] \arrow[dr, tail] & & {[\varphi]} \arrow[uurr, bend left, "u"] \arrow[rr] \arrow[dr, tail] & & 1 \arrow[dr, tail, "\mathtt{t}"] & \\
 & \mathrm{y}d \arrow[rr, "\mathrm{y}f"'] & & \mathrm{y}c \arrow[rr, "\varphi"'] \arrow[uurr, bend left, "(\varphi{,}u)"] & & \mathtt{Cof}
\end{tikzcd}
$$

The above gets simplified when $\Gamma = 1$.

$$c \Vdash [(\varphi, u) : A^+] \qquad \Leftrightarrow \qquad \begin{tikzcd} 1 \arrow[d, "\mathtt{t}"'] & {[\varphi]} \arrow[l] \arrow[r, "u"] \arrow[d, tail, "m"'] \arrow[dl, phantom, very near start, "\urcorner"] \arrow[dr, phantom, very near start, "\ulcorner"] & A \arrow[d, tail, "\eta_A"] \\ \mathtt{Cof} & \mathrm{y}c \arrow[l, "\varphi"] \arrow[r, "{(\varphi,u)}"'] & A^+ \end{tikzcd}$$

Monad structure from dominance

Proposition (Awodey,2018)

$+ : \mathscr{E} \to \mathscr{E}$ *is a (fibred) monad.*

Monad structure from dominance

Proposition (Awodey,2018)

$+ \colon \mathscr{E} \to \mathscr{E}$ is a (fibred) monad.

First, we give a category-theoretic proof.

1st Proof.

η_A, η_{A^+}: cofibrations $\Rightarrow$ $\eta_{A^+} \circ \eta_A$: cofibration by dominance.

η_A: cofibrant partial map classifier $\Rightarrow$ there is a unique morphism μ_A classifying the partial map $(\eta_{A^+} \circ \eta_A, \mathrm{id}_A)$.

$$\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, tail, "\eta"'] \arrow[drr, phantom, "\ulcorner", very near start] & & A \arrow[dd, tail, "\eta"] \\
A^+ \arrow[d, tail, "\eta"'] & & \\
A^{++} \arrow[rr, dashed, "\mu"'] & & A^+
\end{tikzcd}$$

Monad structure from dominance

(1st Proof cont'd.)

μ_A thus obtained is natural in A:

By classifying property of η_B the bottom square commutes since
(i) all vertical squares are pullbacks (I and II because η is cartesian),
(ii) the top square commutes,
(iii) $\eta_{A^+} \circ \eta_A$: cofibration by dominance.

$$
\begin{tikzcd}
& B \arrow[rr, equal] \arrow[dd, tail] && B \arrow[dddd, tail, "\eta_B"] \\
A \arrow[ur, "f"] \arrow[rr, equal] \arrow[dd, tail, "\eta_A"'] \arrow[ddr, phantom, "I"] && A \arrow[ur] \arrow[dddd, tail, "\eta_A"] & \\
& B^+ \arrow[dd, tail, "\eta_{B^+}"] && \\
A^+ \arrow[ur, "f^+"] \arrow[dd, tail, "\eta_{A^+}"'] \arrow[ddr, phantom, "II"] &&& \\
& B^{++} \arrow[rr, "\mu_B"] && B^+ \\
A^{++} \arrow[ur, "f^{++}"] \arrow[rr, "\mu_A"] && A^+ \arrow[ur, "f^+"] &
\end{tikzcd}
$$

Monad structure from dominance

(1st Proof cont'd.)

To see that $\mu \circ \eta_{A^+} = \mathrm{id}_{A^+}$, observe that the following is a pullback by an easy diagram chase using the previous diagram and the fact that η is always monic.

$$\begin{tikzcd}
A \arrow[rr, equal] \arrow[d, tail, "\eta_A"'] \arrow[drr, phantom, very near start, "\ulcorner"] & & A \arrow[d, tail, "\eta_A"] \\
A^+ \arrow[r, "\eta_{A^+}"'] & A^{++} \arrow[r, "\mu"'] & A^+
\end{tikzcd}$$

By the uniqueness of the classifying map of (η_A, id_A), we have $\mu_A \circ \eta_{A^+} = \mathrm{id}_{A^+}$. By naturality of η,

$$\eta_{A^+} \circ \eta_A = (\eta_A)^+ \circ \eta_A$$

The same argument above shows

$$\mu_A \circ \eta_{A^+} = \mathrm{id}_{A^+} \quad .$$

Proposition ((Awodey, 2018))

$+\colon \mathscr{E} \to \mathscr{E}$ *is a (fibred) monad.*

Now, we give a proof using Kripke–Joyal semantics.

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2nd Proof. Write $A^{++} = (A^{+})^{+}$.

$c \Vdash (\varphi, u) : A^{++}$

$\Leftrightarrow$ $u = (\psi, u')$, $c \Vdash [\varphi : \mathtt{Cof}]$, and for every $f \colon c' \to c$, if $c' \Vdash [x : \varphi.f]$ then $c' \Vdash [\psi_f(x) : \mathtt{Cof}]$, and for every $g \colon d \to c'$, if $d \Vdash [y : \psi.g]$ then $d \Vdash [u'_g(y) : A]$ and u' is uniform.

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Now, set $f = \mathrm{id}_c$.

The statement above (after $\Leftrightarrow$) becomes $u = (\psi, u')$ and $c \Vdash \varphi : \texttt{Cof}$, $c \Vdash \psi : [\varphi] \to \texttt{Cof}$, $c \Vdash u' : \sum_{x : [\varphi]} [\psi(x)] \to A$

Monad structure from dominance (II)

The latter implies

$$c \Vdash \mathsf{dom}(\varphi,\psi) : \mathtt{Cof} \text{ and } c \Vdash u' : \mathsf{dom}(\varphi,\psi) \to A.$$

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Uniformity of u' implies $\mathscr{E} \Vdash \mu : A^{++} \to A^+$.
By Yoneda, we get $\mu : A^{++} \to A^+$.

Also, $\mu \circ \eta_{A^+} = \mathrm{id} = \mu \circ +(\eta_A)$ because $\mathrm{dom}(\varphi, \mathtt{t}) = \varphi$ and $\mathrm{dom}(\mathtt{t}, \psi) = \psi$.
$\mu \circ \mu_{A^+} = \mu \circ +(\mu_A)$ because $\mathrm{dom}(\mathrm{dom}(\varphi, \psi), \theta) = \mathrm{dom}(\varphi, \mathrm{dom}(\psi, \theta))$.

An application of Kripke–Joyal semantics: Uniform trivial fibration (I)

For any type A define

$$\mathsf{TFib}(A) := \prod_{\varphi:\mathtt{Cof}} \prod_{u:[\varphi]\to A} \sum_{a:A} u =_{\varphi} a,$$

where the type $u =_{\varphi} a$ (written $(\varphi, u) \nearrow a$ in Orton and Pitts 2018) is defined

$$(u =_{\varphi} a) := \prod_{p:[\varphi]} \mathsf{Eq}_A(up, a)\,.$$

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Proposition

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Recall that p_A being a **a uniform trivial fibration** means that for every cofibration $C \rightarrowtail Z$ and commutative square there is a diagonal filler $j_C(z, a) \colon Z \to \Gamma.A$ making both triangles commute,

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$$\begin{tikzcd}
C' \arrow[r, "f'"] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[r, "a"] \arrow[d, tail] & \Gamma.A \arrow[d, "p_A"] \\
Z' \arrow[r, "f"'] & Z \arrow[r, "z"'] \arrow[ur, dotted, "j_C(z,a)"] & \Gamma
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$$j_{C'}(zf, af') = j_C(z,a) \circ f\ .$$

$$\begin{tikzcd}
C' \arrow[r, "f'"] \arrow[d, tail] \arrow[dr, phantom, very near start, "\ulcorner"] & C \arrow[r, "a"] \arrow[d, tail] & \Gamma.A \arrow[d, "p_A"] \\
Z' \arrow[r, "f"'] \arrow[urr, dotted, "j_{C'}(zf,af')"] & Z \arrow[r, "z"'] \arrow[ur, dotted, "j_C(z,a)"] & \Gamma
\end{tikzcd}$$

Lemma

For $\Gamma \vdash A\,\mathrm{Type}$, $\gamma : \mathrm{y}c \to \Gamma$ *such that*

$$c \Vdash a : A(\gamma)$$
$$c \Vdash \varphi : \mathtt{Cof}(\gamma)$$
$$c \Vdash u : \big([\varphi] \to A\big)(\gamma)\,.$$

then we also have

$$c \Vdash e : (u =_\varphi a)(\gamma) \qquad \Leftrightarrow \qquad \begin{tikzcd} {[\varphi]} \arrow[r, "u"] \arrow[d, tail] & \Gamma.A \arrow[d, "p_A"] \\ \mathrm{y}c \arrow[ur, "\langle\gamma,a\rangle"] \arrow[r, "\gamma"'] & \Gamma \end{tikzcd} \quad \textit{commutes,}$$

where

$$(u =_\varphi a) := \prod_{x:[\varphi]} \mathsf{Eq}_A(ux, a)\,.$$

Proof of Lemma.

$$\begin{tikzcd}
{[\varphi]} \arrow[r, "u"] \arrow[d, tail] & \Gamma.A \arrow[d, "p_A"] \\
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$c \Vdash a : A(\gamma) \Leftrightarrow$ the lower triangle commutes.

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$c \Vdash a : A(\gamma) \;\Leftrightarrow\;$ the lower triangle commutes.

$c \Vdash \varphi : \mathtt{Cof}(\gamma)$ and $c \Vdash \big(u : [\varphi] \to A\big)(\gamma) \;\Leftrightarrow\;$ the outer square commutes.

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$c \Vdash e : u =_\varphi a(\gamma)$
$\Leftrightarrow\; c \Vdash e : \prod_{x:[\varphi]} \mathrm{Eq}_A(ux, a)(\gamma)$
$\Leftrightarrow\;$ for all $f : d \to c$ in $\mathcal{C}$, $d \Vdash x : [\varphi](\gamma.f)$ returns $d \Vdash e_f(x) : \mathrm{Eq}_A(ux, a)(\gamma.f)$
$\Leftrightarrow\;$ the top triangle commutes. QED.

Proof of Theorem.

Suppose $\Gamma \vdash \alpha : \mathsf{TFib}(A)$.

Thus for all $\gamma \colon \mathrm{y}c \to \Gamma$, we have $c \Vdash \alpha_\gamma : \mathsf{TFib}(A)(\gamma)$, coherently in γ.

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Note that

$$\begin{aligned} \mathrm{TFib}(A) &= \prod_{\varphi:\mathtt{Cof}} \prod_{u:[\varphi]\to A} \sum_{a:A} \prod_{x:[\varphi]} \mathrm{Eq}_A(ux, a) \\ &= \prod_{(\varphi,u)\,:\,A^+} \sum_{a:A} u =_\varphi a \end{aligned}$$

We thus obtain

$$c \Vdash \alpha_\gamma : \prod_{(\varphi,u)\,:\,A^+} \sum_{a:A} (u =_\varphi a)(\gamma)\ .$$

Proof of Theorem (cont'd).

By Kripke–Joyal semantics of $\prod$ and $\sum$, we have for every $f : d \to c$ in $\mathcal{C}$, if

$$d \Vdash (\varphi, u) : A^{+}(\gamma.f) \tag{1}$$

then

$$d \Vdash \alpha_{\gamma.f}(\varphi, u)^{0} : A(\gamma.f) \tag{2}$$

and

$$d \Vdash \alpha_{\gamma.f}(\varphi, u)^{1} : \left(u =_{\varphi} \alpha_{\gamma.f}(\varphi, u)^{0}\right)(\gamma.f) \tag{3}$$

and, for any $g : d' \to d$,

$$\alpha_{\gamma.f}(\varphi, u).g = \alpha_{(\gamma.fg)}(\varphi[g], u[g])\,. \tag{4}$$

An application of Kripke–Joyal semantics: Uniform trivial fibration (III)

Unfolding the condition (1) yields the following commutative diagram.

$$
\begin{tikzcd}
{[\varphi.f]} \arrow[r, "{\langle \gamma.f, u_f \rangle}"] \arrow[d, tail] & \Gamma.A \arrow[d, "p_A"] \\
\mathrm{y}d \arrow[r, "\gamma.f"'] & \Gamma
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\end{tikzcd}
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Lemma applied to (2) and (3) yields the following commuting diagram.

$$
\begin{tikzcd}
\varphi.f \arrow[r, "{\langle \gamma.f, u_f \rangle}"] \arrow[d, tail] & \Gamma.A \arrow[d, "p_A"] \\
\mathrm{y}d \arrow[r, "\gamma.f"'] \arrow[ur, "{\alpha_{\gamma.f}(\varphi, u)^0}" description] & \Gamma
\end{tikzcd}
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An application of Kripke–Joyal semantics: Uniform trivial fibration (III)

Thus forcing $\mathsf{TFib}(A)$ produces diagonal fillers

$$j_\varphi(\gamma, u) \triangleq \alpha_{\gamma.f}(\varphi, u)^0$$

for each lifting problem as in the right hand square below:

$$\begin{tikzcd}
\varphi[f] \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & {[\varphi]} \arrow[rr, "{\langle\gamma,u\rangle}"] \arrow[d, tail] \arrow[dr] & & \Gamma.A \arrow[dd, "p_A"] \\
& & 1 \arrow[dd, "\mathtt{t}", near end] & \\
\mathrm{y}d \arrow[r, "\mathrm{y}f"'] \arrow[uurrr, dashed, bend left, "{\alpha_{\gamma.f}(\varphi[f],u[f])^0}" description] \arrow[drr, bend right, "{\varphi[f]}"'] & \mathrm{y}c \arrow[rr, "\gamma"'] \arrow[dr, "\varphi"'] \arrow[uurr, dashed, bend right, "{\alpha_\gamma(\varphi,u)^0}" description] & & \Gamma \\
& & \mathtt{Cof} &
\end{tikzcd}$$

Proof of Theorem (cont'd) – converse argument

If $p_A \colon \Gamma.A \to \Gamma$ is a uniform trivial fibration then in particular for every *basic* cofibration $[\varphi] \rightarrowtail \mathrm{y}c$ and square as on the right below, there is a diagonal filler $j_\varphi(\gamma, u)$ as indicated.

$$\begin{tikzcd}
{[\varphi.f]} \arrow[r] \arrow[d, tail] & {[\varphi]} \arrow[r, "u"] \arrow[d, tail] & \Gamma.A \arrow[d, "p_A"] \\
\mathrm{y}c' \arrow[r, "\mathrm{y}f"'] \arrow[urr, dotted, "j_{\varphi.f}(\gamma.f, u.f)"] & \mathrm{y}c \arrow[r, "\gamma"'] \arrow[ur, dotted, "j_\varphi(\gamma, u)"] & \Gamma
\end{tikzcd}$$

Proof of Theorem (cont'd) – converse argument

If $p_A : \Gamma.A \to \Gamma$ is a uniform trivial fibration then in particular for every *basic* cofibration $[\varphi] \rightarrowtail \mathrm{y}c$ and square as on the right below, there is a diagonal filler $j_\varphi(\gamma, u)$ as indicated.

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\mathrm{y}c' \arrow[r, "\mathrm{y}f"'] \arrow[urr, dotted, "j_{\varphi.f}(\gamma.f, u.f)"] & \mathrm{y}c \arrow[r, "\gamma"'] \arrow[ur, dotted, "j_\varphi(\gamma, u)"'] & \Gamma
\end{tikzcd}
$$

By the lemma, this corresponds to an element $\alpha_\gamma : \mathrm{y}c \to \mathsf{TFib}(A)$ over $\gamma : \mathrm{y}c \to \Gamma$,

$$
\begin{tikzcd}
& \Gamma.\mathsf{TFib}(A) \arrow[d, "p_{\mathsf{TFib}(A)}"] \\
\mathrm{y}c \arrow[r, "\gamma"'] \arrow[ur, dotted, "\alpha_\gamma"] & \Gamma
\end{tikzcd}
$$

Proof of Theorem (cont'd) – converse argument

The uniformity condition says exactly that for all $f : c' \to c$, the elements α_γ cohere, $\alpha_{(\gamma.\mathrm{y}f)} = \alpha_\gamma \circ f$.

Proof of Theorem (cont'd) – converse argument

The uniformity condition says exactly that for all $f : c' \to c$, the elements α_γ cohere, $\alpha_{(\gamma.\mathrm{y}f)} = \alpha_\gamma \circ f$.
By Yoneda for the slice category $\mathscr{E}/\Gamma$ that there is a term $\Gamma \vdash \alpha : \mathsf{TFib}(A)$. QED.

Further Applications: A fibrant universe of fibrant types

As in (Orton and Pitts, 2018), we assume a tiny interval I in $\mathscr{E}$ equipped with the following structures:

- Terms $0, 1 : I$,
- *connections* $\_ \sqcap \_ ,\ \_ \sqcup \_ : I \times I \to I$,

satisfying certain axioms.

Further Applications: A fibrant universe of fibrant types

Using the amazing right adjoint $(-)^I \dashv \sqrt[I]{(-)}$ and the forcing of the type

$$\mathtt{Fill}_e(A) \;=\; \prod_{\varphi:\mathtt{Cof}} \; \prod_{u:[\varphi]\to\prod_{i:I} Ai} \; \prod_{a_e:Ae} (ue =^{\varphi} a_e) \to \sum_{a:\prod_{i:I} Ai} (ae =_{Ae} a_e) \times (u =^{\varphi} a)\,,$$

we show that that the universe $\mathtt{Fib}$ of fibrant types is itself fibrant.

$$\begin{tikzcd}
\mathtt{Fib}(A) \arrow[r] \arrow[d] & \mathtt{Fib} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner" very near start] & & \sqrt[I]{\mathrm{Type}_\bullet} \arrow[d] \\
\Gamma \arrow[u, dashed, bend left] \arrow[r, "A"'] & \mathrm{Type} \arrow[r, "\eta"'] & \sqrt[I]{\mathrm{Type}^I} \arrow[r, "\sqrt[I]{\mathtt{Fill}}"'] & \sqrt[I]{\mathrm{Type}}
\end{tikzcd}$$

Next ...

Further use of Kripke–Joyal semantics for dependent type theory in

- Showing Frobenius property of fibrations.
- Showing equivalence extension property.
- Extending the semantics to sheaf toposes
- Studying equivariant fibrations by means of Kripke–Joyal semantics in topos of group actions.

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The End

Thanks for your attention!