Are Monoidal Fibrations Instances Of 2-fibrations?

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Abstract

In these notes I will relate the notion to monoidal fibration introduced in (Shulman 2009) to the notion of 2-fibration in (Hermida 1999), (Bakovic 2012) and (Buckley 2014).

According to (Shulman 2009):

Definition *0.1.* Suppose $(\mathcal{X}, \otimes, k)$ and $(\mathcal{C}, \otimes', k')$ are monoidal categories. A **monoidal fibration** is a functor $P\colon \mathcal{X} \to \mathcal{C}$ such that that

1. P is a Grothendieck fibration,
2. P is a strict monoidal functor, and
3. The tensor product $\otimes$ of $\mathcal{X}$ preserves cartesian arrows.

If P is also an opfibration and $\otimes$ preserves opcartesian arrows, we say that P is a **monoidal bifibration**.

In a monoidal fibration with cartesian base, each fibre is monoidal and each transition functor f^* is strong monoidal. Shulman calls the monoidal structure on $\mathcal{X}$ the **external** monoidal structure, and the monoidal structures on fibres the **internal** monoidal structures.

Construction *0.2* (fibrewise/internal monoidal structure). Let $P\colon (\mathcal{X}, \otimes, k) \to (\mathcal{C}, \times, 1)$ be a cloven monoidal fibration, and let $B \in \mathcal{C}$. We define a monoidal structure on the fiber $\mathcal{X}_B$ as follows. The unit object is $k_B = !_B^* k$, and the tensor product is given by

$$X \boxtimes Y = \Delta_B^*(X \otimes Y)$$

where $X, Y \in \mathcal{X}_B$.

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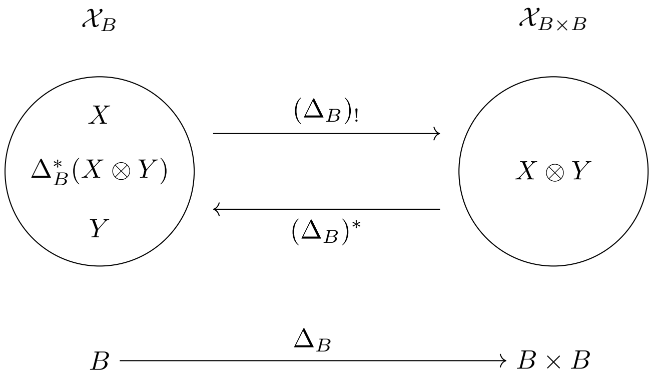


EXAMPLE *0.3*. A cloven fibration $(\mathrm{cod}, \rho) : \mathcal{C}^{[1]} \to \mathcal{C}$ is precisely a category $\mathcal{C}$ with a choice of pullbacks in $\mathcal{C}$. The fibre over Γ is the slice category $\mathcal{C}/\Gamma$, and base change functors are pullback functors. Then fibrewise/internal tensor product in $\mathcal{C}/\Gamma$ is fibre product (aka pullback): if $p\colon X \to \Gamma$, and $q\colon Y \to \Gamma$, then $X \boxtimes Y = X \times_\Gamma Y$, and $p \boxtimes q = \Delta^*(p \times q)$ since

$$\begin{array}{ccc} X \times_\Gamma Y & \longrightarrow & X \times Y \\ {\scriptstyle p\boxtimes q}\downarrow & \lrcorner & \downarrow{\scriptstyle p\times q} \\ \Gamma & \xrightarrow{\Delta} & \Gamma \times \Gamma \end{array}$$

We want to relate monoidal fibration structure of P to 2-fibration structure of delooping functor $\mathbb{P}$ of P. First, we recall the definition of strict 2-fibration from (Hermida 1999) and weak 2-fibration from (Bakovic 2012) and (Buckley 2014):

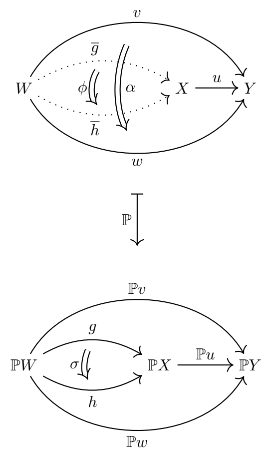
Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor. Inspired by the case of 1-functors we define 2-cartesian 1-cells as follows.

DEFINITION *0.4*. A 1-cell $u\colon X \to Y$ in $\mathbb{X}$ is **cartesian** with respect to $\mathbb{P}$ whenever for each 0-cell W in $\mathbb{X}$ the following commuting square is a (strict) pullback diagram in 2-category $\mathfrak{Cat}$.

$$\begin{array}{ccc} \mathbb{X}(W,X) & \xrightarrow{u_*} & \mathbb{X}(W,Y) \\ {\scriptstyle \mathbb{P}_{W,X}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathbb{P}_{W,Y}} \\ \mathbb{C}(\mathbb{P}W,\mathbb{P}X) & \xrightarrow[\mathbb{P}(u)_*]{} & \mathbb{C}(\mathbb{P}W,\mathbb{P}Y) \end{array}$$

REMARK *0.5*. By considering object component of pullback diagram above we observe that every 2-cartesian 1-cell is automatically 1-cartesian in the usual sense.

This definition gives us two layers of cartesian properties of 1-cells w.r.t. $\mathbb{P}$ in $\mathbb{X}$. First of all, u is 1-cartesian as usual. Second, every 2-cell $\alpha\colon v \Rightarrow w\colon W \to Y$ and every 2-cell $\sigma\colon g \Rightarrow h\colon \mathbb{P}W \to \mathbb{P}X$ with $\mathbb{P}(\alpha) = \mathbb{P}(u) \centerdot \sigma$ there is a unique lift ϕ of σ such that $u \centerdot \phi = \alpha$.



Definition *0.6*. A 2-functor $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a ***2-fibration*** if

1. any 1-cell in $\mathbb{C}$ of the form $f\colon A \to \mathbb{P}X$ has a 2-cartesian lift,
2. $\mathbb{P}$ is a local fibration, that is the functor $\mathbb{P}_{X,Y}\colon \mathbb{X}(X,Y) \to \mathbb{C}(\mathbb{P}X,\mathbb{P}Y)$ is a Grothendieck fibration for every pair of objects X, Y in $\mathbb{X}$, and
3. cartesian 2-cells in $\mathbb{X}$ are closed under pre-composition and post-composition with arbitrary 1-cells.

Now, suppose $P\colon (\mathcal{X},\otimes,k) \to (\mathcal{C},\otimes',k')$ is monoidal fibration. The fact that P is a strict monoidal functor, makes it possible to define a (strict) delooping 2-functor $\mathbb{P}\colon \mathbb{B}\mathcal{X} \to \mathbb{B}\mathcal{C}$ of P. The condition of local fibration of $\mathbb{P}$ corresponds exactly to the fact that P is fibration of categories. The third condition in definition (0.6) says that cartesian morphisms in $\mathcal{X}$ are preserved under tensoring. Now, we want to see what condition in definition (0.6) says. First of all it gives, for every object A of $\mathcal{C}$, a lift $\widetilde{A}$ in $\mathcal{X}$ such that $\widetilde{A}$ is cartesian, that is given objects X in $\mathcal{X}$ and B in $\mathcal{C}$ with $A \otimes' B = P(X)$, there is a unique object $\bar{B}$ in $\mathcal{X}$ such that $\widetilde{A} \otimes \bar{B} = X$. Moreover, given any morphisms $f\colon X_0 \to X_1$ and a morphism $g\colon B_0 \to B_1$ with $A \otimes' g = P(f)$ there is a unique morphism $\bar{g}\colon \bar{B}_0 \to \bar{B}_1$ such that $\widetilde{A} \otimes \bar{g} = f$.

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URL: *arXiv:0706.1286*