Some algebraic geometric

k: field (alg closed, character zero)

let $\Gamma_n = k[x_1, \ldots, x_n]$

Γ_n is a k-algebra. (commutative and unital)

Suppose A: k-alg

$$\text{k-alg}(\Gamma_n, A) \cong A^n = \{(a_1, \ldots, a_n) \mid a_i \in A\}$$

$$\Gamma_n \xrightarrow{\varphi} A \longmapsto \varphi(x_1, \ldots, x_n)$$

Lemma. ~~If $f = g_1 f_1 + \cdots + g_n f_n$~~

~~$g_i \in \Gamma_n$ and $f_i \in \Gamma_n$~~

~~then common zeros of f~~

A: k-alg

$I \subset k[x_1, \ldots, x_n]$ ideal

Then define $V_I(A) = \{\vec{x} \in A^n \mid f(\vec{x}) = 0 \;\; \forall f \in I\}$

Prop. $V_I : k\text{-alg}^{op} \longrightarrow \mathrm{Set}$ is a representable functor and is represented by T_n/I

$$V_I(A) \cong k\text{-alg}\left(\frac{k[x_1,\dots,x_n]}{I}, A\right)$$

$$(\varphi(x_1),\dots,\varphi(x_n)) \longleftarrow\!\shortmid \varphi : \frac{k[x_1,\dots,x_n]}{I} \to A$$

for any $f \in I$

$$f(\varphi(x_1),\dots,\varphi(x_n)) =$$

$$\sum c_{(\alpha_1,\dots,\alpha_n)} \varphi(x_1)^{\alpha_1} \cdots \varphi(x_n)^{\alpha_n} =$$

$$\varphi\left(\sum c_{(\alpha_1,\dots,\alpha_n)} x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) = \quad \text{b/c } \varphi : k\text{-alg mor.}$$

$$\varphi(f(x_1,\dots,x_n)) = 0_A$$

$$\begin{tikzcd} \mathrm{Psh}(k\text{-alg}) = [k\text{-alg}^{op}, \mathrm{Set}] \\ \\ k\text{-alg}^{op} \arrow[uu, "\mathrm{Spec} = \text{yoneda}"] \end{tikzcd}$$

$A : k\text{-alg}$

$$\mathrm{Spec}_A : (k\text{-alg}) \longrightarrow \mathrm{Set}$$

$$\mathrm{Spec}_A \, \xi := k\text{-alg}(A, \ast)$$

$$\cancel{\mathrm{Spec}_A \, \xi \, k[x_1, \ldots, x_n]/I \cong V_I(A)}$$

$$\mathrm{Spec}_{T_n}(B) = k\text{-alg}(T_n, B) = V_I(B)$$

<u>Defn.</u> Scheme is a presheaf which is locally like Spec.

We can take k to be a comm. unital ring.

Remark A, B, C: (unital) Commutative K-algebras.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[dl, phantom, "\lrcorner"] & b \arrow[d, mapsto] \\ C \arrow[r] & B \otimes_A C & b\otimes 1 \\ c \arrow[r, mapsto] & 1\otimes c & \end{tikzcd}$$

is a pushout of K-algebras.

Consider B and C as A-modules. Construct

$B \otimes_A C = \dfrac{\text{Free}(B \times C)}{\sim}$ $\quad (f(a).b, c) \sim (b, g(a).c)$

so $(f(a).b) \otimes c = b \otimes (g(a).c)$

If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[ddr, bend left, "\lambda"] & \\ C \arrow[r] \arrow[drr, bend right, "\gamma"'] & B\otimes_A C \arrow[dr, dashed, "\exists! u"] & \\ & & D \end{tikzcd}$$

If $\lambda f = \gamma g$

$\exists!\ u : B\otimes_A C \longrightarrow D$

s.t. $u(b\otimes 1) = \lambda(b)$

$u(1\otimes c) = \gamma(c)$

Construct $u : B \otimes_A C \longrightarrow D$ defined on

basic tensors $\quad b\otimes c \longmapsto \lambda(b).\gamma(c)$

It is well defined b/c $\lambda f = \gamma g$.

$u(f(a).b \otimes c) = \lambda(f(a).b).\gamma(c) = \lambda f(a).\lambda(b).\gamma(c)$

$= \lambda(b).\gamma g(a).\gamma(c) = u(b \otimes g(a).c) =$

Take $f \in A$: k-alg

$$
\begin{tikzcd}
k[x] \arrow[r, "\text{loc.}"] \arrow[d, "f"'] & k[x,x^{-1}] \cong \frac{k[x,y]}{\langle xy-1\rangle} \arrow[d] \\
f \in A \arrow[r, "\text{loc.}"] \arrow[d, "h"'] & A_f \arrow[d] \\
B \arrow[r, "\text{loc.}"] & A_{h(f)} = A_{h\circ f}
\end{tikzcd}
$$

pushout of k-algebras

$$\sum c_i x^i + \sum d_j y^j \mapsto \sum c_i f^i + \sum \frac{d_j}{f^j}$$

(ex)

$$
\begin{tikzcd}
k[x] \arrow[r] \arrow[d] & k[x,x^{-1}] \arrow[d] \\
A \arrow[r] \arrow[d] & A_f \arrow[d] \\
A_g \arrow[r] & A_f \otimes_A A_g
\end{tikzcd}
$$

So $A_f \otimes_A A_g \cong A_{fg}$

Remark. Every commutative ring R is a comm. $\mathbb{Z}$-algebra.

So coproduct and pushout of comm. rings can be given as ~~coproduct~~ coproduct & pushout of comm. comm. $\mathbb{Z}$-algebras.

A, B, C : rings

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\
C \arrow[r] & B \otimes_A C
\end{tikzcd}
$$

$\mathbb{Z}$: initial comm. ring

and coproduct

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & B \otimes_{\mathbb{Z}} C
\end{tikzcd}
$$

Defn. A monoidal monad S on monoidal category $(\mathcal{C}, \otimes, k, a, \ell, r)$ is a monad (S, η, μ) on category $\mathcal{C}$ plus ~~morphisms~~ natural tranformation

$$\tau_{-,-} : S(- \otimes -) \Rightarrow S(-) \otimes S(-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

compatibility

and a morphism

$$\tau_k : S(k) \longrightarrow k$$

s.t.

$$\begin{tikzcd}
SX \otimes Sk \arrow[d, "1 \otimes \tau_k"'] \arrow[dr, phantom, "="] & S(X \otimes k) \arrow[l, "\tau_{X,k}"'] \arrow[d, "S(r_X)"] \\
SX \otimes k \arrow[r, "r_{S(X)}"'] & S(X)
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
Sk \otimes SX \arrow[d, "\tau_k \otimes 1"'] \arrow[dr, phantom, "="] & S(k \otimes X) \arrow[l, "\tau_{k,X}"'] \arrow[d, "S(\ell_X)"] \\
k \otimes SX \arrow[r, "\ell_{SX}"'] & S(X)
\end{tikzcd}$$

$$r_{S(X)} \circ (1 \otimes \tau_k) \circ \tau_{X,k} = S(r_X) \qquad \ell_{SX} \circ (\tau_k \otimes 1) \circ \tau_{k,X} = S(\ell_X)$$

and

- Comptability of η ~~and~~ with τ_k:

$$\begin{tikzcd} k \arrow[d, "\eta"'] \arrow[dr, "\eta"] & \\ S(k) \arrow[r, "\tau"'] \arrow[ur, phantom, "="] & k \end{tikzcd}$$

- Comptability of η ~~and~~ with $\tau_{X,Y}$:

$$\begin{tikzcd} X\otimes Y \arrow[d, "\eta_{X\otimes Y}"'] \arrow[dr, "\eta\otimes\eta"] & \\ S(X\otimes Y) \arrow[r, "\tau"'] \arrow[ur, phantom, "="] & S(X)\otimes S(Y) \end{tikzcd}$$

- Compatibility of μ with τ_k:

$$\begin{tikzcd} S^2(k) \arrow[r, "S(\tau_k)"] \arrow[d, "\mu_k"'] \arrow[dr, phantom, "="] & S(k) \arrow[d, "\tau_k"] \\ S(k) \arrow[r, "\tau_k"'] & k \end{tikzcd}$$

- Compatibility of μ with $\tau_{X,Y}$

$$\begin{tikzcd} S^2(X\otimes Y) \arrow[r, "\mu_{X,Y}"] \arrow[d, "S(\tau_{X,Y})"'] & S(X\otimes Y) \arrow[dd, "\tau_{X,Y}"] \\ S(SX\otimes SY) \arrow[d, "\tau_{SX,SY}"'] \arrow[r, phantom, "="] & \\ S^2X\otimes S^2Y \arrow[r, "\mu_X\otimes\mu_Y"'] & SX\otimes SY \end{tikzcd}$$

Prop.

Let S be a monoidal monad on a tensor category $(\mathcal{C}, \otimes, k, a, l, r)$.

Then the category $Alg(S)$ of S-algebras is again a tensor category.

Proof. Structure maps of S: $\eta, \mu, \tau_{X,Y}, \tau_k$

where

$$\eta_X : X \longrightarrow S(X)$$

$$\mu_X : S^2(X) \longrightarrow S(X)$$

$$\tau_{X,Y} : S(X \otimes Y) \longrightarrow S(X) \otimes S(Y)$$

$$\tau_k : S(k) \longrightarrow k$$

$\tilde{k} = (k, \tau_k : S(k) \to k)$ defines an

S-algebra:

- Compatibility of η with τ_k

$$\begin{tikzcd} k \arrow[d, "\eta_k"'] \arrow[dr, "\eta"] \arrow[d, phantom, shift left=4ex, "="] & \\ S(k) \arrow[r, "\tau_k"'] & k \end{tikzcd}$$

and

- Compatibility of η with τ_k

$$\begin{tikzcd} S^2(k) \arrow[r, "S(\tau_k)"] \arrow[d, "\mu_k"'] \arrow[dr, phantom, "="] & S(k) \arrow[d, "\tau_k"] \\ S(k) \arrow[r, "\tau_k"'] & k \end{tikzcd}$$

Tensor product on $\underbrace{}_{}$ $\mathrm{Alg}(S)$

$\tilde{A} = (A, \alpha : S(A) \to A)$

$\tilde{B} = (B, \beta : S(B) \to B)$

$$\tilde{A} \otimes \tilde{B} := \left(A \otimes B,\ \begin{tikzcd} S(A\otimes B) \arrow[rr] \arrow[dr, "\tau_{A,B}"'] & & A \otimes B \\ & SA \otimes SB \arrow[ur, "\alpha\otimes\beta"'] & \end{tikzcd}\right)$$

Check $\bar{A}\otimes\bar{B}$ is indeed an $\mathcal{S}$-algebra.

$$
\begin{tikzcd}
A\otimes B \arrow[d, "\eta_{A\otimes B}"'] \arrow[dr, "=\eta_A\otimes\eta_B"] \arrow[drr, "1=1\otimes 1"] & & \\
\mathcal{S}(A\otimes B) \arrow[r, "\tau_{A,B}"'] & \mathcal{S}A\otimes\mathcal{S}B \arrow[r, "\alpha\otimes\beta"'] & A\otimes B
\end{tikzcd}
$$

we used ∘ compatibility of η with $\tau_{X,Y}$.

$\cancel{\tau\circ\eta}$

$(\tau\circ\eta=\eta\otimes\eta)$

$$
\begin{tikzcd}
\mathcal{S}^2(A\otimes B) \arrow[r, "\mathcal{S}(\tau_{A,B})"] \arrow[dd, "\mu_{A\otimes B}"'] & \mathcal{S}(\mathcal{S}A\otimes\mathcal{S}B) \arrow[r, "\mathcal{S}(\alpha\otimes\beta)"] \arrow[d, "\tau_{SA,SB}"'] \arrow[dr, phantom, "\substack{\text{naturality of}\\ = \ \tau_{X,Y}}"] & \mathcal{S}(A\otimes B) \arrow[d, "\tau_{A,B}"] \\
\arrow[r, phantom, "\substack{=\\ \text{Compatibility}\\ \text{of } \mu \text{ and}\\ \tau_{X,Y}}"] & \mathcal{S}^2A\otimes\mathcal{S}^2B \arrow[r, "\cancel{\mathcal{S}\alpha\otimes\mathcal{S}\beta}"] \arrow[d, "\mu_A\otimes\mu_B"] \arrow[dr, phantom, "\substack{\langle\alpha,\beta\rangle:\text{algebras}\\ =}"] & \mathcal{S}A\otimes\mathcal{S}B \arrow[d, "\alpha\otimes\beta"] \\
\mathcal{S}(A\otimes B) \arrow[r, "\tau_{A,B}"'] & \mathcal{S}A\otimes\mathcal{S}B \arrow[r, "\alpha\otimes\beta"'] & A\otimes B
\end{tikzcd}
$$

$\tilde{k} = (k, \mathcal{S}(k) \xrightarrow{\tau_k} k)$ is the unit of tensor in $\mathrm{Alg}(\mathcal{S})$.

~~This corresponds~~

$\tilde{A} = (A, \mathcal{S}(A) \xrightarrow{\alpha} A)$

$$\tilde{k} \otimes \tilde{A} = \left(k \otimes A,\ \begin{tikzcd} \mathcal{S}(k\otimes A) \arrow[rr] \arrow[dr, "\tau_{k,A}"'] & \arrow[d, phantom, "="] & k \otimes A \\ & \mathcal{S}k \otimes \mathcal{S}A \arrow[ur, "\tau_k \otimes \alpha"'] & \end{tikzcd} \right)$$

$\tilde{k} \otimes \tilde{A} \cong \tilde{A}$

$$\begin{tikzcd} \mathcal{S}(k\otimes A) \arrow[rr, "\mathcal{S}(l_A)"] \arrow[d, "\tau_{k,A}"'] & & \mathcal{S}(A) \arrow[dd, "\alpha"] \\ \mathcal{S}k \otimes \mathcal{S}A \arrow[r, "\tau_k \otimes 1"] \arrow[d, "\tau_k \otimes \alpha"'] & k \otimes \mathcal{S}A \arrow[ur, "l_{\mathcal{S}(A)}"] \arrow[dl, "1 \otimes \alpha"] & \\ k \otimes A \arrow[rr, "l_A"'] & & A \end{tikzcd}$$

compatibility of $\tau_{X,Y}$, τ_k $\mathcal{S}$

b/c of naturality of $l_{(-)}$

$$\begin{tikzcd} \mathrm{Alg}(S) \arrow[d, "U_S"'] & \tilde{A} = (A,\ S(A) \xrightarrow{\alpha} A) \arrow[d, mapsto] \\ \mathcal{C} & A \end{tikzcd}$$

$$U_S(\tilde{k}) = k.$$

$$U_S(\tilde{A} \otimes \tilde{B}) = U_S(\tilde{A}) \otimes U_S(\tilde{B}) = \cancel{U_S}(A \otimes B$$

U_S is a strict monoidal functor.

$$\left\{ \begin{array}{c} \text{monoidal} \\ \text{monads} \\ \text{on} \\ \mathcal{C} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{monoidal} \\ \text{structures on} \\ \mathrm{Alg}(S) \text{ which} \\ \text{make } U_S \text{ a} \\ \text{strict monoidal} \\ \text{functor.} \end{array} \right\}$$

Prop.

The category CjSLat of complete join semi-lattices is symmetric closed monoidal.

Defn: If M, N, L are sup-lattices ~~and~~ then $f : M \times N \to L$ is a ~~bimorphism~~ bihomomorphism of sup-lattices if f preserves suprema in each variable i.e.

$$f\Big(\bigvee_{i \in I} x_i , y\Big) = \bigvee_{i \in I} f(x_i, y)$$

and

$$f\Big(x , \bigvee_{j \in J} y_j\Big) = \bigvee_{j \in J} f(x, y_j)$$

In CjSLat $M \otimes N$ is the codomain of universal bimorphism

$$\begin{tikzcd} M \times N \arrow[r, "{[\,]}"] \arrow[dr, "f"'] & M \otimes N \arrow[d, dashed, "\exists ! \bar{f}"] \\ & L \end{tikzcd}$$

$M \otimes N$ can be obtained as a quotient of free sup-lattice of $M \times N$ (which is $P(M \times N)$) by the equivalence relation generated by

$$\bigvee_{i \in I} x_i \otimes y \sim \Big(\bigvee_{i \in I} x_i\Big) \otimes y \quad \text{and}$$

$$\bigvee_{j \in J} (x \otimes y_j) \sim x \otimes \Big(\bigvee_{j \in J} y_j\Big)$$

$$M \otimes N = \frac{P(M \times N)}{\sim}$$

<u>Note</u>: In particular

$$0 = 0 \otimes y$$
$$0 = x \otimes 0$$

$\forall x, y$.

Unit of tensor on CjSLat

free (complete) sup-lattice on one generator

$P1 = \{\bot \leq \top\}$

CjSLat	AbGrp
$\frac{P(M\times N)}{\sim} = M\otimes N$	$A\otimes B = \frac{Fr_{Ab}(A\times B)}{\sim}$
free sup-lattice on one-generator $= P1$	$\mathbb{Z}$ = free abelian group on one generator
$\bigvee_{i\in I} a_i$	$\sum_{i\in I:\text{finite}} a_i$

$P1\otimes M \cong M$, $M\otimes P1 \cong M$

$$\begin{aligned} P1\otimes M &\xrightarrow{\cong} M \\ \bot\otimes m &\longmapsto 0 \\ \top\otimes m &\longmapsto m \end{aligned}$$

Rmk: M^{op}: opposite poset (category)

$$\mathrm{Hom}(M, (P1)^{op}) \cong \mathrm{Hom}(P1, M^{op})$$

$$\cong M^{op}$$

internal hom. Closed structure.

$$\mathrm{Hom}(M,N) \cong \mathrm{Hom}(N^{op}, M^{op})$$

$$\cong \mathrm{Hom}(N^{op}, \mathrm{Hom}(P1, M^{op}))$$

$$\cong \mathrm{Hom}(N^{op}, \mathrm{Hom}(M, P1^{op}))$$

$$\cong \mathrm{Hom}(N^{op} \otimes M, P1^{op})$$

$$\cong (N^{op} \otimes M)^{op}$$

Also

$$M \otimes N \cong \mathrm{Hom}(M, N^{op})^{op}$$

tensor is given by Hom.

Rmk. Q SLat $(\mathcal{S})$ where

$\mathcal{S}$ is any elementary

topos.

In that case

unit is $P1 \cong \Omega$

subobject classifier.

You can do this internal to every topos $\mathcal{S}$. ⟶ k-$\mathbb{Z}$-monad

$1 = * = \{pt\}$

$(Set, \times, *) \circlearrowleft P$

P: power set monad

algebras of P $\cong$ sup-lattices

structure map $\cong$ structures map

$(X, \alpha : PX \longrightarrow X)$

α is a join.

$$\begin{array}{c} Alg(P) \\ \Big\downarrow U_P \\ (\mathcal{S}, \times, *) \circlearrowleft P \end{array}$$

Note that U_P is not a

Guess fibres of U_P = propositions. (at best) or contractible.

Symmetric algebra monad:

Idea: for a vector space $V/_k$,

$SV =$ free commutative algebra over V

Construction: $V/_k$: a vector space over a field k.

The symmetric algebra SV is generated by elements of V using operations:

(i) addition and scalar multiplication

(ii) an associative binary operation $\circ$

$V/_k \longrightarrow$ Consider V as a set

$\downarrow$

Consider algebra generated by V

$x \in V,\ y \in V \longmapsto x + y \in \langle V \rangle$

$x \in V,\ \lambda \in k \longmapsto \lambda x \in \langle V \rangle$

$x \in V,\ y \in V \longmapsto x \circ y \in V$ subjet to

+ $(x \circ y) \circ z = x \circ (y \circ z)$
+ $(\lambda x) \circ (\lambda' y) = (\lambda \lambda')(x \circ y)$
+ $v \circ w = w \circ v$

Prop.

SV is a commutative graded algebra

spanned by p-fold products, that is

elements of the form

$$V^{\otimes p} = \{ v_1 \cdots v_p \} \cong \{ v_1 \otimes \cdots \otimes v_p \}$$

$$S_p \times V^{\otimes p} \longrightarrow V_p^{\otimes p}$$

$$(\pi, v_1 \cdots v_p) \longmapsto \prod_{i=1}^{p} v_{\pi(i)}$$

More generally,

Suppose $(\mathcal{C}, \otimes, k)$ a symm. monoidal category. w/ countable coproducts

$V \in \mathcal{C}$

Form tensor powers $V^{\otimes n}$

and their countable coproduct

$$TV = \bigoplus_{n \geq 0} V^{\otimes n} \in \mathcal{C}$$

Q: Is TV a monoid object in $\mathcal{C}$?

Yes, if the tensor product distributes over (these) coproducts

$k \to TV = \bigoplus_{n \geq 0} V^{\otimes n}$ is

just embedding of a summand.

$TV \otimes TV \xrightarrow{m} TV$ is got by

$$\begin{array}{c} TV \otimes TV \\ \shortparallel \\ \left(\bigoplus_{m\geq 0} V^{\otimes m} \otimes \bigoplus_{n \geq 0} V^{\otimes n}\right) \end{array} \xrightarrow{\cong} \bigoplus_{\substack{m \geq 0 \\ n \geq 0}} V^{\otimes m} \otimes V^{\otimes n} \xrightarrow{\cong} \bigoplus_{\substack{m\geq 0 \\ n \geq 0}} V^{\otimes (m+n)} \to TV$$

Action of symmetric group S_n

$S_n \times \boxed{V^{\otimes n}} \longrightarrow V^{\otimes n}$

$\cancel{\text{each}}\ \boxed{S_n} \longrightarrow \boxed{\mathcal{C}(V^{\otimes n}, V^{\otimes n})}$

$\pi \longmapsto V^{\otimes n} \longrightarrow V^{\otimes n}$

$V \otimes V \otimes \cdots \otimes V \longrightarrow V \otimes \cdots \otimes V$

$\mathcal{C}$: linear category

$$\mathcal{S}_n \longrightarrow \mathcal{C}(V^{\otimes n}, V^{\otimes n})$$

$$\sigma \longmapsto \hat{\sigma}$$

define $\quad P_A : V^{\otimes n} \longrightarrow V^{\otimes n}$

$$P_A = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \hat{\sigma}$$

$$P_A \cdot P_A \in \mathcal{C}(V^{\otimes n}, V^{\otimes n})$$

$$P_A \cdot P_A = \frac{1}{n!} \frac{1}{n!}$$

$$\mathcal{C}(V^{\otimes n}, V^{\otimes n}) \times \mathcal{C}(V^{\otimes n}, V^{\otimes n}) \longrightarrow \mathcal{C}(V^{\otimes n}, V^{\otimes n})$$

$$\hat{\sigma} \qquad \hat{\sigma} \longmapsto \hat{\sigma}^2 = \hat{\sigma}$$

$$\hat{\sigma}\hat{\tau} = \widehat{\sigma\tau}$$

$$P_A^2 = \frac{1}{n!}\frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \tau \in S_n}} \widehat{\sigma\tau}$$

I have a group G of size n.

$$\frac{1}{n}\sum_{g\in G} g = \boxed{avg(G)}$$

$$\frac{1}{n}\frac{1}{n}\sum_{\substack{g\in G\\ h\in G}} gh = avg(G) \qquad avg_2(G)$$

$$\frac{1}{n^2}\left(\begin{array}{l} \cancel{g_0(e+g\ldots)}\ (e+g_1 \cdots\cdots g_{n-1}) \\ \quad\quad\quad\ (e+g_1+\cdots+g_{n-1}) \\ g_1 \\ \quad\quad\quad\quad\quad \vdots \end{array}\right)$$

$\cancel{avg(G)}$

$$\sum_{g\in G} g \quad \sum_{g\in G} g \quad \cdots\cdots \quad \sum_{g\in G} g$$

$$\frac{1}{n^2}\left(n^2\cdot avg(G) = \right) = avg(G)$$

e.g.

$$S_2 \longrightarrow \mathcal{L}(V^{\otimes 2}, V^{\otimes 2})$$

$$1 \longmapsto \hat{1}\,(v_1 \otimes v_2) = v_1 \otimes v_2$$

$$\sigma \longmapsto \hat{\sigma}\,(v_1 \otimes v_2) = v_2 \otimes v_1$$

$$P_A = \frac{1}{2!}(\hat{1} + \hat{\sigma})$$

$$P_A(v_1 \otimes v_2) = \frac{1}{2} v_1 \otimes v_2 + \frac{1}{2} v_2 \otimes v_1 =$$

$$\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$$

$\hat{1}\hat{1} =$
$\hat{\sigma}\hat{\sigma} =$
$\hat{\sigma}\hat{1} =$
$\hat{1}\hat{\sigma} =$

$$P_A\left(\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)\right) =$$

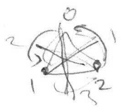
$$\frac{1}{2!}(\hat{1} + \hat{\sigma})\left(\frac{v_1 \otimes v_2 + v_2 \otimes v_1}{2}\right) =$$

$$\frac{1}{4}(\boxed{v_1 \otimes v_2} + v_2 \otimes v_1) + \frac{1}{4}(v_2 \otimes v_1 + v_1 \otimes v_2)$$

$$\frac{1}{2}(v_1 \otimes v_2) + \frac{1}{2}(v_2 \otimes v_1)$$

$$\frac{1}{3!}(\hat{1} + \hat{\mu} + \hat{\mu}^2 + \hat{\rho}_1 + \hat{\rho}_2 + \hat{\rho}_3)$$

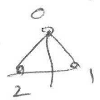
$v_1 \otimes v_2 \otimes v_3$ $\quad$ $v_1 \otimes v_2 \otimes v_3$



$$\frac{1}{6}(v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_1 \otimes v_2 + v_2 \otimes v_3 \otimes v_1 +$$

$$(v_2 \otimes v_1) \otimes v_3 + v_1 \otimes v_3 \otimes v_2 + v_3 \otimes v_2 \otimes v_1) =$$

$(v_2 \otimes v_1) \otimes v_3$ $\quad$ $v_1 \otimes v_2 \otimes v_3$



(9)

$p_A : V^{\otimes n} \longrightarrow V^{\otimes n}$

$p_A^2 = p_A$

If idempotent splits in $\mathcal{C}$, we can form its cokernel:

Remark

$$A \overset{e}{\underset{1}{\rightrightarrows}} A \dashrightarrow B$$

is a coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "e"] \arrow[r, shift right, "1"'] & A \arrow[r, "r"] \arrow[dr, "f"'] & B \arrow[d, dashed, "fs=x"] \\ & & C \end{tikzcd}$$

$fe = f$

$fsr = f$

$xr = f$

~~$xr = xrsr = xre = fe = fsr$~~

$x = xrs = fs$

$v_1 \otimes \cdots \otimes v_n \longmapsto [v_1 \otimes \cdots \otimes v_n]$

$$V^{\otimes n} \underset{1}{\overset{\rho_A}{\rightrightarrows}} V \overset{\pi}{\dashrightarrow} \mathcal{S}V$$

and

$$\mathcal{S}V = \bigoplus_{n \geq 0} S^n V$$

SV = free commutative monoid object in $(\mathcal{C}, \otimes, k)$

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & SV \arrow[d, dashed] \\ & A \end{tikzcd}$$

: comm. monoid.

<u>Prop.</u> $\mathcal{S}_\bullet$ - algebras are exactly k- commutative unital algebras.

$$
\begin{tikzcd}
\mathrm{CjSLat} \arrow[d, shift left, "\text{forgetful} = U_1"] \\
\mathrm{jSLat} \arrow[u, bend left, "F_1 = \text{free}"] \arrow[u, phantom, "\dashv"] \arrow[d, shift left, "\text{forgetful} = U_0"] \\
\mathrm{Set} \arrow[u, bend left, "F_0 = \text{free}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

$X : \mathrm{Set}$

$F_0(X)$ = set of finite subsets of X

$(F_0(X), \cup, \emptyset)$

$$
\begin{tikzcd}
X \arrow[r, hook, "x \mapsto \{x\}"] \arrow[dr, "f"'] & F_0(X) \arrow[d, "\bar{f}"] \\
& A
\end{tikzcd}
$$

A: join-semilattice

$\bar{f}$: unique extension of f to a morphism of join semi-lattices

$\underset{\text{finite}}{\Gamma \subseteq X} \qquad \bar{f}(\Gamma) = \bigvee_{\gamma \in \Gamma} f(\gamma) \in A.$

$$
\begin{tikzcd}
\mathrm{CjSlat} \arrow[d, "U_1"] \\
\mathrm{jSlat} \arrow[u, bend left, "F_1"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

$A : \mathrm{jSlat}$

$F_1(A) := IA$ = set of ideals of A.

~~Lemma.~~

$$a \longmapsto \downarrow a$$

$$A \hookrightarrow IA$$

faithful (one-to-one)

$(IA, \vee, \{0\})$

where $I \vee J = \{ i \vee j \mid i \in I, j \in J \}$

$$\downarrow a \vee \downarrow b = \downarrow (a \vee b)$$

$\bar{g}(\downarrow a) := g(a)$

$\bar{g}(I) = \bar{g}\left(\bigvee_{a \in I} \downarrow a\right)$

$= \bigvee_{a \in I} g(a)$

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "g"'] & IA \arrow[d, dotted, "\bar{g}"] \\
& C
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\mathrm{Alg}(F), \otimes, F1) \arrow[d] \\
(\mathrm{Set}, \times, 1) \arrow[loop right, "F"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(\mathrm{Alg}(P), \otimes, P1) \arrow[d] \\
(\mathrm{Set}, \times, 1) \arrow[loop right, "P"]
\end{tikzcd}
$$

F: finit power set
monoidal monad

P: power set
monoidal
monad

$\mathrm{Alg}(F) \cong j\mathrm{SLat}$

$\mathrm{Alg}(P) \cong Cj\mathrm{SLat}$