

Relative Pseudo-Monads

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Motivation:

- Bicategory of pro-functors

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	Traditional	Kleisli	Relative
1-dim	Standard Notion of Monad (Triple)	Kleisli Triple (Monads '76)	Altenkirch et al. ('13)
2-dim	Pseudo-monads (Bunge '74)	No-iteration of Pseudo-monad (Marmolejuk Wood '14)	Relative Pseudo-monads

Q: any formulation like

$\mathcal{P}t \xrightarrow{\text{law}} \mathcal{K} \ ?$

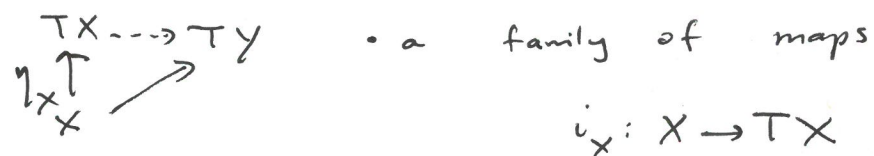
for pseudo-m.

# Kleisli Triple:

Def.

- a function  $T: \text{ob}(\mathbb{C}) \rightarrow \text{ob}(\mathbb{C})$
- a family of functions  $\llbracket x \circ y \rrbracket$

$$(-)^{\dagger}: \mathbb{C}(X, TY) \rightarrow \mathbb{C}(TX, TY)$$



satisfying

$$(1) (g^{\dagger} f)^{\dagger} = g^{\dagger} f^{\dagger}$$

$$X \xrightarrow{f} TY \xrightarrow{g^{\dagger}} TZ$$

$$Y \xrightarrow{g} TZ$$

$$\begin{array}{ccc} X & \xrightarrow{i_x} & TX \xrightarrow{f^{\dagger}} TY \\ & & \parallel \\ & & X \xrightarrow{f} TY \end{array}$$

Define  
 $T(f) = (f^{\dagger})^{\dagger}$   
 $f: X \rightarrow Y$

$$(2) f = f^{\dagger} i_x$$

$$(3) (i_x)^{\dagger} = \top_{TX}$$

## Example (Power Set)

$$X \longmapsto PX = \{ S \mid S \text{ subset of } X \}$$

$$X \xrightarrow{f} PY \rightsquigarrow f^{\dagger}: PX \rightarrow PY$$

$$S \longmapsto \bigcup_{x \in S} f(x)$$

$$\begin{array}{ccc} X & \xrightarrow{i_x} & PX \\ x & \longmapsto & \{x\} \end{array}$$

2. Kleisli category

Given  $(T, (-)^{\dagger}, i)$  define

$Kl(T)$  as

(i) obj: Same  $\mathcal{C}$

(ii) maps:  $\text{Hom}_{Kl(T)}(X, Y) = \mathcal{C}(X, TY)$

(iii) composition

$$\frac{X \xrightarrow{f} TY \quad Y \xrightarrow{g} TZ}{X \xrightarrow{g^{\dagger} \circ f} TZ}$$

①

②

Kleisli  $\longrightarrow$  Relative Monads

} generalizes  
to level 2

Relative Pseudomonads

③

② Relative Monad

Fix a category  $\mathcal{C}$  and inclusion  $J: \mathcal{C} \rightarrow \mathcal{D}$

(a subcategory)

Def. A relative monad over  $J$  &

• a function  $T: \text{ob}(\mathbb{C}) \rightarrow \text{ob}(\mathbb{D})$

• a family of functions

$$(-)^{\dagger}: \mathcal{D}(JX, TY) \rightarrow \mathcal{D}(TX, TY)$$

• a family of maps  $i_X: JX \rightarrow TX$

satisfying:

Same equations

$$\begin{array}{ccc} JX & \xrightarrow{i_X} & TX \\ & \searrow f & \downarrow f^{\dagger} \\ & & TY \end{array}$$

There is a way to  
develop a theory of EM-algebras;

Ex. Set  $\xrightarrow{J}$  Classes

$$X \longmapsto P(X) = \{S \mid S \subseteq X\}$$

More in "Constructive Set theory".

Also, related to universes in type theory  
when you do not have power type  
but embedding to other larger universes

Ex. (power set)  $P \mapsto \text{Kl}(P) \subseteq \text{Rel}$   
 functor  
 $P$

Def

Kleis: cat of relative monads.

lim  $(T, \langle \_ \rangle, i)$  Def in  $\text{Kl}(T)$

obj: same as  $\mathcal{C}$

mor:  $\text{Hom}_{\text{Kl}(T)}(X, Y) := \text{ID}(TX, TY)$

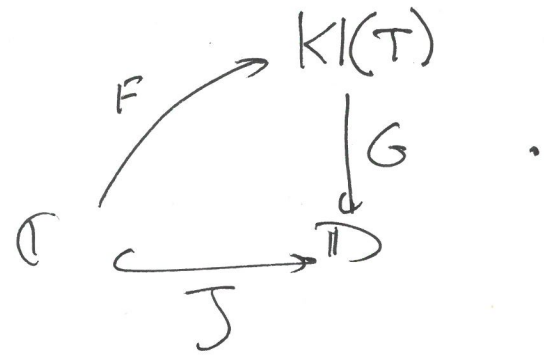
Comp: 
$$\begin{array}{ccc} TX \xrightarrow{f} TY & & TY \xrightarrow{g} TZ \\ \hline TX \xrightarrow{f} TY \xrightarrow{g} TZ \end{array}$$

identities  $TX \xrightarrow{i_X} TX$

"Specification"

$J: \mathcal{C} \longrightarrow \mathcal{D}$

$\text{Rel. Mnd}(\mathcal{C}) \xrightleftharpoons[\text{extension}]{\text{restriction}} \text{Mnd}(\mathcal{D})$



Example of "Relative adjunctions"

defined only on some objects.

③

Relative Pseudomonads

Fix 2 categories  $\mathbb{C} \xrightarrow{J} \mathbb{D}$   
inclusion

(intuitive example)

$J: \text{Cat}_{\text{small}} \rightarrow \text{CAT}$   
Small Cats      loc. small cats

Def. A relative pseudomonad

- a functor  $T: \text{ob}(\mathbb{C}) \rightarrow \text{ob}(\mathbb{D})$
- a family of functors  $(\ )^{\dagger}: \mathbb{D}(Jx, Ty) \rightarrow \mathbb{D}(Tx, Ty)$
- a family of maps  $\Leftrightarrow i_x: Jx \rightarrow Tx$

PLUS:

$x \in \mathbb{C}$