Relative Pseudo-Monads

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Motivation:

- Bicategory of pro-functors

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	Traditional	Kleisli	Relative
1-dim	Standard Notion of Monad ~~(triple)~~	Kleisli Triple (Manes '76)	Altenkirch et al. ('13)
2-dim	Pseudo-monads	No-iteration of Pseudo-monad	Relative Pseud-monads
	(Bunge '74)	(Marmolejo & Wood '14)	

Q: any formulation like

$pt \xrightarrow[\text{lax}]{} k$?

for pseudo-cs

Kleisli Triple:

Def. • a function $T: ob(\mathbb{C}) \to ob(\mathbb{C})$

• a family of functions "$x \otimes y$"

$$(-)^{+}: \mathbb{C}(X, TY) \to \mathbb{C}(TX, TY)$$

$$\begin{tikzcd} TX \arrow[r, dashed] & TY \\ X \arrow[u, "\eta_X"] \arrow[ur] \end{tikzcd}$$

• a family of maps

$$i_X : X \to TX$$

satisfying

(1) $(g^{+} f)^{+} = g^{+} f^{+}$

$$X \xrightarrow{f} TY \xrightarrow{g^{+}} TTZ$$

$$Y \xrightarrow{g} TZ$$

$$X \xrightarrow{i_X} TX \xrightarrow{f^{+}} TY$$
$$\quad\quad\quad\, ||$$
$$\quad\quad\quad X \xrightarrow{f} TY$$

Define $T(f) = (i_Y f)^{+}$

$f: X \to Y$

(2) $f = f^{+} i_X$

(3) $(i_X)^{+} = 1_{TX}$

Example (Power Set)

$X \mapsto PX = \{ S \mid S$ \subset \part $\subseteq X\}$

$$X \xrightarrow{f} PY \rightsquigarrow f^{+}: PX \to PY$$
$$S \mapsto \bigcup_{x \in S} f(x)$$

$$X \xrightarrow{i_X} PX$$
$$x \mapsto \{x\}$$

2. Kleisli category

Given $(T, (-)^\dagger, i)$ define

$Kl(T)$ as

(i) obj: same $\mathbb{C}$

(ii) maps: $\mathrm{Hom}_{Kl(T)}(X,Y) = \mathbb{C}(X, TY)$

(iii) composition
$$\frac{X \xrightarrow{f} TY \qquad Y \xrightarrow{g} TZ}{X \xrightarrow{g^\dagger \circ f} TZ}$$

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1 Kleisli $\longrightarrow$ 2 Relative Monads

generalizing to level 2

Relative Pseudomonads 3

2 Relative Monad.

Fix a category $\mathbb{C}$ and inclusion $J : \mathbb{C} \hookrightarrow \mathbb{D}$ (a subcategory)

Def. A relative monad over J &

- a function $T : ob(\mathbb{C}) \longrightarrow ob(\mathbb{D})$

- a family of functions

$$(-)^{\dagger} : \mathbb{D}(JX, TY) \longrightarrow \mathbb{D}(TX, TY)$$

- a family of maps $i_X : JX \longrightarrow TX$

satisfying:

Same equations

$$\begin{tikzcd} JX \arrow[r, "i_X"] \arrow[dr, bend right, "f"'] & TX \arrow[d, "f^{\dagger}"] \\ & TY \end{tikzcd}$$

There is a way to develope a theory of EM-algebras;

Ex. $\quad \mathrm{Set} \xrightarrow{J} \mathrm{Classes}$

$$X \longmapsto P(X) = \{S \mid S \subseteq X\}$$

More in "Constructive Set theory".

Also, related to universe in Type theory when you do not have power type but embedding to other larger universes

Ex. (powerset functor P) $\quad P \rightsquigarrow Kl(P) \subseteq Rel$

Def

Kleisli cat of Relative monads,

Given $(T, (-)^{+}, i)$ Define $Kl(T)$

obj: same as $\mathbb{C}$

mor: $Hom_{Kl(T)}(X, Y) := \mathbb{D}(JX, TY)$

Comp:

$$\frac{JX \xrightarrow{f} TY \qquad JY \xrightarrow{g} TZ}{JX \xrightarrow{f} TY \xrightarrow{g^{+}} TZ}$$

identities $\quad JX \xrightarrow{i_X} TX$

"Speculation"

$J : \mathbb{C} \longmapsto \mathbb{D}$

$$Rel.Mnd(\mathbb{C}) \underset{\text{extension}}{\overset{\text{restriction}}{\rightleftarrows}} Mnd(\mathbb{D})$$

$$\begin{tikzcd} & \mathrm{Kl}(T) \arrow[d, "G"] \\ \mathbb{C} \arrow[ur, bend left, "F"] \arrow[r, hook, "J"'] & \mathbb{D} \end{tikzcd}$$.

Example of "Relative adjunction"

defined only on some objects.

(3)

Relative Pseudo-monads

Fix 2-categories $\mathbb{C} \overset{J}{\hookrightarrow} \mathbb{D}$ inclusion

(intuitive example $J : \underset{\text{small Cats}}{\mathrm{Cat}} \hookrightarrow \underset{\text{loc. small Cats}}{\mathrm{CAT}}$

Def. A relative pseudomonad

- a function $T : ob(\mathbb{C}) \longrightarrow ob(\mathbb{D})$
- a family of functors $(\ )^{\dagger} : \mathbb{D}(JX, TY) \to \mathbb{D}(TX, TY)$
- a family of maps $\forall\ i_X : JX \to TX$, $x \in \mathbb{C}$

PLUS: