Some notes on Ω_j in presheaf toposes

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Abstract

As we discussed given a local operator[1] $j : \Omega \to \Omega$ in a topos $\mathcal{E}$ one can obtain a retract Ω_j of Ω by taking equalizer of id and j. We remarked that both Ω and Ω_j are injective objects of $\mathcal{E}$ and Ω_j is precisely the image of j. In these notes, I will show how to obtain the object Ω_j in $\mathcal{E} \simeq Psh(\mathcal{C}, \mathbb{J})$ and j is the classifying morphism of subobject $\mathbb{J} \hookrightarrow \Omega$. My reference for some of the material here is (MacLane & Moerdijk 1992), particularly section 5.1.

1 Closed sieves

Suppose S is a sieve on object U of category $\mathcal{C}$. For every composable pair of morphisms $f : V \to U$ and $g : W \to V$, we have

$$\Updownarrow \frac{S \ni fg}{f^*S \ni g}$$

By putting $g = id_V$, we have $f \in S$ if and only if f^*S is the maximal sieve $\mathcal{C} \downarrow V$. Furthermore, given a Grothendieck topology[2] $\mathbb{J}$ on $\mathcal{C}$ we say a sieve S on U covers a morphism $f : V \to U$ if $f^*S \in \mathbb{J}(V)$. We denote this by $S \blacktriangleright f$.[3] We then have[4]

$$\Updownarrow \frac{S \blacktriangleright f \circ g}{f^*S \blacktriangleright g} \tag{1}$$

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[1] Also known as Lawvere-Tierney topology.

[2] A better name is coverage.

[3] This covering relation is mentioned implicitly without the notation in chapters 2 and 3 of (MacLane & Moerdijk 1992). My feeling is that by making it explicit not only will there be less verbiage but also it will be useful for forming derivation trees of proofs. Perhaps a fancier way of defining the covering relation $\blacktriangleright$ is to say it is a subfunctor of product functor $\langle Sub \circ y, \mathbf{el} \circ y \rangle : \mathcal{C} \to \mathfrak{Cat} \times \mathfrak{Cat}^{\mathrm{op}}$ where $\mathbf{el} : Psh(\mathcal{C}) \to \mathfrak{Cat}$ assign to each presheaf on $\mathcal{C}$ its category of elements.

[4] The form of this derivation suggests that there must some adjunction between suitable categories lurking around.

for every morphism $g\colon W \to V$. Notice that $S \blacktriangleright id_U$ iff S covers U iff $S \in \mathbb{J}(U)$. Moreover, if $S \blacktriangleright id_U$ then $S \blacktriangleright f$ for any f with $\mathrm{cod}(f) = U$. Also, any sieve S covers any of its members: $S \ni f \implies S \blacktriangleright f$.

Remark 1.1. Stability condition of Grothendieck topology $\mathbb{J}$ can be formulates as

$$\Downarrow \frac{S \blacktriangleright f}{S \blacktriangleright f \circ g} \tag{2}$$

and transitivity is to say for any sieve R on U

$$\Downarrow \frac{S \blacktriangleright f \text{ and } R \blacktriangleright S}{R \blacktriangleright f} \tag{3}$$

where $R \blacktriangleright S$ is short for $R \blacktriangleright f$ for all $f \in S$.

Corollary 1.2. Since we always have $R \ni f \implies R \blacktriangleright f$ for any sieve R on U and any morphisms $f\colon V \to U$, from (3) above we conclude that

$$\Downarrow \frac{S \blacktriangleright f \text{ and } S \subset R}{R \blacktriangleright f} \tag{4}$$

Definition 1.3. A sieve M on object U is ***closed*** whenever for any morphisms $f : V \to U$, if $M \blacktriangleright f$ then $M \ni f$.

Remark 1.4. By (1), pulling back a closed sieve gives a closed sieve. That is,

$$\Downarrow \frac{M : \text{closed}}{f^{*}M : \text{closed}}$$

Example 1.5. Take the unit circle $\mathbb{T}$ represented by $\{z \in \mathbb{C} : ||z|| = 1\}$ where $\mathbb{C}$ is the complex plane. Take the subspace topology on $\mathbb{T}$ inherited from Euclidean topology on $\mathcal{C}$. Choose opens $U_0 = \{e^{it} : -0.00001 < t < \pi\}$ and $U_1 = \{e^{it} : \pi < t < 2\pi + 0.00001\}$. Then $\downarrow U_0 \cup \downarrow U_1$ is a sieve on $\mathbb{T}$ which is not closed. If we tweak U_0 and U_1 a bit we get U_0' and U_1' as the upper and lower open semicircles. Then the sieve $\downarrow U_0 \cup \downarrow U_1$ is closed but not covering.

1.1 Closure of sieves

For any sieve S on U we want to construct the smallest closed sieve $\overline{S}$ on U which contains S. The characteristic morphism of $\mathbb{J} \hookrightarrow \Omega$, j, is given by $j_U(S) = \{h\colon \centerdot \to U | S \blacktriangleright h\}$. We use closure operator associate to j to define $\overline{S}$; form the pullback

$$\begin{array}{ccc} \overline{S} & \xrightarrow{\;!\;} & 1 \\ \downarrow & \lrcorner & \downarrow \\ yU & \xrightarrow[j \circ \chi_S]{} & \Omega \end{array}$$

So,

$$\begin{aligned}\overline{S}(V) &= \{h : V \to U \mid j(h^*S) = \text{maximal sieve on V}\}\\ &= \{h : V \to U \mid h^*S \blacktriangleright 1_V\}\\ &= \{h : V \to U \mid S \blacktriangleright h\}\end{aligned}$$

Therefore, $\overline{S} = \{h \mid \mathrm{cod}(h) = U \text{ and } S \blacktriangleright h\}$. For start notice that S covers its own members, So, indeed $S \subset \overline{S}$. Stability condition of Grothendieck topology implies that $\overline{S}$ is a sieve. Following derivation shows that $\overline{S}$ is closed.

$$\cfrac{\cfrac{\overline{S} \blacktriangleright f \qquad S \blacktriangleright \overline{S}}{S \blacktriangleright f}}{\overline{S} \ni f}$$

Any inclusion $S \hookrightarrow M$ of S into a closed sieve factors through $S \hookrightarrow \overline{S}$. Following derivation exhibits this fact:

$$\begin{array}{c} \overline{S} \ni f \\ \hline S \blacktriangleright f \\ \hline M \blacktriangleright f \\ \hline M \ni f \end{array}$$

Exercise 1.6. Closure operation commutes with restriction (pullback) operation, that is for any morphism $g\colon V \to U$ in $\mathcal{C}$, $g^*(\overline{S}) = \overline{g^*S}$

2 Defining Ω_j in presheaf toposes

Suppose a site $(\mathcal{C}, \mathbb{J})$ is given. In the topos $\mathcal{E} = Psh(\mathcal{C}, \mathbb{J})$ the subobject classifier is computed by $\Omega(U) = \{\text{Sieves on } U\}$. Define Ω_j to be presheaf obtained as the equalizer

$$\Omega_j \overset{e}{\rightarrowtail} \Omega \underset{id}{\overset{j}{\rightrightarrows}} \Omega$$

We then have for every U in $\mathcal{C}$,

$$\begin{aligned}\Omega_j(U) &= \{\text{Sieves } S \text{ on } U \mid j_U(S) = S\}\\ &= \{\text{Sieves } S \text{ on } U \mid \{V \xrightarrow{h} U : S \blacktriangleright h\} = S\}\\ &= \{\text{Closed sieves } S \text{ on } U\}\end{aligned}$$

Ω_j acts on morphism by restriction (i.e. pulling back) of closed sieves. See (1.4). More explicitly $\Omega_j(f)(M) = f^*M$.

First we show that Ω_j is $\mathbb{J}$-separated presheaf and ultimately a $\mathbb{J}$-sheaf.

Proposition 2.1. Ω_j just defined is separated presheaf, meaning that there is at most one way to amalgamate any compatible family of local sections to get a global section.

Proof. Let U be any object of $\mathcal{C}$ and S a covering sieve on U. Take two global sections M and N in $\Omega_j(U)$ and suppose they yield equal local sections on S. Thus we have two closed sieves M and N on U with $f^*M = f^*N$ for any $f \in S$. In particular $M \cap S = N \cap S$ as sieves on U. We prove that $M = N$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{M \ni f}{M \blacktriangleright f} \qquad S \blacktriangleright f}{M \cap S \blacktriangleright f}}{N \cap S \blacktriangleright f}}{N \blacktriangleright f}}{N \ni f}$$

□

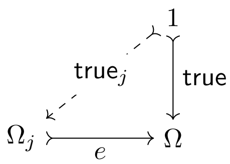
Proposition 2.2. Ω_j just defined is a sheaf, meaning that there is exactly one way to amalgamate any compatible family of local sections to get a global section.

Proof. What we need to show is existence of such amalgamation. Consider an object U in $\mathcal{C}$ and a sieve S on U. Take any compatible family $\{M_\rho \in \Omega_j(\mathrm{dom}(\rho)) : \rho \in S\}$ of closed sieves indexed by elements of sieve S. The natural thing to do is to combine all of these closed sieves to obtain a sieve on U. So, we form $M = \{\rho \circ g : g \in M_\rho\}$. One easily sees that M is indeed a sieve on U. However, it need not be closed. Again the natural thing to do is to consider its closure $\overline{M} = \{h : . \to U : M \blacktriangleright h\}$. It is a closed sieve with right amalgamation property, that is $\rho^*\overline{M} = M_\rho$ for any ρ in S. To show this one only needs to prove that $\rho^*M = M_\rho$ because of (1.6). $M_\rho \subset \rho^*M$ is true by definition of M. For other way around, we have:

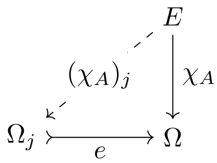
$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\rho^*M \ni f}{M \ni \rho f}}{\exists \rho' \in S\ \exists f' \in M_{\rho'}\ .\ \rho' f' = \rho f}}{(f')^*M_{\rho'} = f^*M_\rho}}{\text{maximal sieve on } \mathrm{dom}(f) = f^*M_\rho}}{M_\rho \blacktriangleright f}}{M_\rho \ni f}$$

□

Remark 2.3. The subobject classifier morphism $\mathtt{true} : 1 \rightarrowtail \Omega$ factors through inclusion $\Omega_j \rightarrowtail \Omega$ simply because maximal sieves are closed.



Suppose a subobject $A \rightarrowtail E$ in topos $\mathcal{E}$ is characterized by the morphism (or predicate if you like) $\chi_A : E \to \Omega$. Then its closure $\overline{A} \rightarrowtail E$ is characterized by $j \circ \chi_A$. A is closed iff $j \circ \chi_A = \chi_A$, which is to say χ_A factors through Ω_j:



Thus we have isomorphism

$$\phi \colon \mathit{Hom}_{\mathcal{E}}(E, \Omega_j) \cong \mathit{ClSub}_{\mathcal{E}}(E) \colon \psi \tag{5}$$

natural in E; where $\phi(a) = (ea)^*(\mathsf{true})$ and $\psi(A) = (\chi_A)_j$. Let us examine this isomorphism for representable presheaves:

$$\Omega_j(U) \cong \mathit{Hom}_{\mathcal{E}}(yU, \Omega_j) \cong \mathit{ClSub}(yU) \cong \{ \text{ closed sieves on } U \}$$

> This of course establishes Ω_j as ***closed subobject classifier*** of $\mathcal{E}$. We know what Ω_j looks like in $\mathit{Psh}(\mathcal{C}, \mathbb{J})$, and now we want to know what closed subsheaves of a given presheaf are. Here is something to keep in mind: Sub-presheaves of sheaves need not be sheaves themselves. For instance take sheaf C_0 of real valued continuous functions on real line $\mathbb{R}$ with its standard Euclidean topology. Presheaf BC_0 of real-valued bounded continuous functions on $\mathbb{R}$ is a subobject of C_0 in $\mathit{Psh}(\mathcal{O}(\mathbb{R}), \mathbb{J}_{\mathbb{R}})$.

Lemma 5.1.4 (MacLane & Moerdijk 1992) proves that in any elementary topos $\mathcal{E}$ equipped with a local operator $j \colon \Omega \to \Omega$ and $A \rightarrowtail E$ any subobject of a j-sheaf E, A is a j-sheaf iff A is a closed subobject of E. We would like to give a more down-to-earth description of a closed sub-presheaf of a presheaf. Notice that by discussion above we have that A is a closed sub-presheaf of presheaf E if and only if its classifying natural transformation $\chi_A \colon E \to \Omega$ factors through Ω_j. This means for any U in $\mathcal{C}$ and any local section $x \in E(U)$, sieve $\chi_A(U)(s) = \{ \xrightarrow{f} U |\ x \centerdot f \in A(\mathrm{dom}(f))\}$ is closed. This is equivalent to the following rule:

$$\Downarrow \frac{\{ \xrightarrow{g} V | x \centerdot fg \in A(\mathrm{dom}(g))\} \in \mathbb{J}(V)}{x \centerdot f \in A(V)} \tag{6}$$

for any $f\colon V \to U$. Setting $f = id$, we get:

$$\Downarrow \frac{\{ \xrightarrow{g} U | x \centerdot g \in A(\mathrm{dom}(g))\} \in \mathbb{J}(U)}{x \in A(U)}$$

This leads to the following corollary:

Corollary 2.4. Let E be as sheaf on a site $(\mathcal{C}, \mathbb{J})$, and A a sub-presheaf of E. A is a sheaf iff for every object U of $\mathcal{C}$ and every section $x \in E(U)$ and for every covering sieve S of U, we have $x \in A(U)$ whenever $x \centerdot f \in A(V)$ for every $f\colon V \to U$ in S.

We can also compute the closure of any sub-presheaf $A \hookrightarrow E$:

$$\begin{aligned}
\overline{A}(U) &= \{x \in E(U) |\ j\chi_A(U)(x) = \text{maximal sieve on } U\} \\
&= \{x \in E(U) |\ j(\{ \xrightarrow{f} U |\ x \centerdot f \in A(U)\}) = \text{maximal sieve on } U\} \\
&= \{x \in E(U) |\ \{V \xrightarrow{g} U | \{ \xrightarrow{f} U |\ x \centerdot f \in A(U)\} \blacktriangleright g\} = \text{maximal sieve on } U\} \\
&= \{x \in E(U) |\ \{g |\ \{W \xrightarrow{k} V |\ x \centerdot gk \in A(W)\} \in \mathbb{J}(V)\} = \text{maximal sieve on } U\} \\
&= \{x \in E(U) |\ \{ \xrightarrow{k} U |\ x \centerdot k \in A(\mathrm{dom}(k))\} \in \mathbb{J}(U)\} \\
&= \{x \in E(U) |\ \exists S \in \mathbb{J}(U) \forall f \in S,\ x \centerdot f \in A(\mathrm{dom}(f))\}
\end{aligned}$$

Example 2.5. Every locale L has a canonical Grothendieck topology $\mathbb{J}_{can}$. This gives rise to local operator in $Psh(L)$. Suppose both $A \rightarrowtail E$ in $Psh(L)$. $\overline{A}$ is again a presheaf and is given by:

$$\overline{A}(u) = \{x \in E(U) |\ \exists \{u_i\}_{i \in I} \text{ in } L \text{ such that } u = \bigvee u_i \text{ and } \forall i.\ x|_{u_i} \in A(u_i)\}$$

Example 2.6. Take locale of real numbers $\mathbb{R}$. Consider presheaf BC_0 of real-valued bounded continuous functions as subpresheaf of the sheaf C_0 of real-valued continuous functions. Then, for instance $tan(\pi x) \in \overline{BC_0}(\frac{-1}{2}, \frac{1}{2})$ but $tan(\pi x) \notin BC_0(\frac{-1}{2}, \frac{1}{2})$. Indeed, $\overline{BC_0} = C_0$

3 A bigger picture of things

A bigger picture I got from reading (MacLane & Moerdijk 1992) is as follows: Start with an elementary topos $\mathcal{E}$ equipped with a local operator j. Our example of his elementary topos in these notes was $Psh(\mathcal{C}, \mathbb{J})$ and j the locale operator associated to the Grothendieck topology $\mathbb{J}$. One defines a j-sheaf by saying an object F of $\mathcal{E}$ is a j-sheaf whenever every dense[5] subobject $m\colon A \rightarrowtail E$ in $\mathcal{E}$, pre-composition with m induces a bijection of sets

[5] m is dense if $\overline{m} = 1_E$.

$Hom_{\mathcal{E}}(E, F) \to Hom_{\mathcal{E}}(A, F)$. Then one shows category of j-dense objects form a subtopos $Sh_j(\mathcal{E})$ where the lex left adjoint to embedding is sheafification functor $a\colon \mathcal{E} \to Sh_j(\mathcal{E})$.

By classifying property of Ω in $\mathcal{E}$ we have:

$$Hom_{\mathcal{E}}(E, \Omega) \cong Sub_{\mathcal{E}}(E)$$

natural in object E. We also mentioned that Ω_j is a closed subobject classifier in $\mathcal{E}$:

$$Hom_{\mathcal{E}}(E, \Omega_j) \cong ClSub_{\mathcal{E}}(E)$$

Moreover by the fact that Ω_j is a j-sheaf and using adjunction $a \dashv inc$, we have:

$$Sub_{Sh_j(\mathcal{E})}(aE) \cong Hom_{Sh_j(\mathcal{E})}(aE, \Omega_j) \cong Hom_{\mathcal{E}}(E, \Omega_j) \cong ClSub_{\mathcal{E}}(E)$$

In particular, if E is a sheaf then $Sub_{Sh_j(\mathcal{E})}(E) = ClSub_{\mathcal{E}}(E)$. In topos of presheaves this implies that a sub-presheaf of a sheaf is a sheaf itself iff it is a closed sub-presheaf.

References

MacLane, S. & Moerdijk, I. (1992), 'Sheaves in geometry and logic', *Springer-Verlag New York* .