Notes on Locally Cartesian Closed Categories

Sina Hazratpour

13/05/2019

Contents

1 Locally cartesian closed categories

We begin by having some insights into slice and comma categories.

Construction 1.1. Suppose $\mathcal{C}$ is category. For an object I of $\mathcal{C}$ the **slice category** $\mathcal{C}/I$ is defined as comma category $(\mathrm{Id}_{\mathcal{C}} \downarrow X)$: we denote the objects of $\mathcal{C}/I$ by tuples $(X, p \colon X \to I)$.

Proposition 1.2. *1. The monomorphisms is the slice category $\mathcal{C}/I$ are exactly the monomorphisms in $\mathcal{C}$.*

2. A morphism $e \colon (Y, q) \to (X, p)$ is an epimorphisms is the slice category $\mathcal{C}/I$ if it is an epimorphism in the category $\mathcal{C}$. However, the converse is not true. Therefore, the forgetful functor $\mathrm{dom}\, \mathcal{C}/I \to \mathcal{C}$ does not preserve epimorphisms, and hence colimits.

Proof. 1. Suppose $m \colon (A, f) \to (B, g)$ is a monomorphism in the slice category $\mathcal{C}/I$. Assume $a, a' \colon X \to A$ such that $ma = ma'$. Therefore, $fa = gma = gma' = fa'$. Hence, $a, a' \colon (X, fa) \to (A, f)$ is a morphism in $\mathcal{C}/I$ with $ma = ma'$ and by our supposition we have $a = a'$. The converse is obvious.

2. The implication is obvious. A counter-example for the converse statement is constructed as follows: Consider the category $\mathcal{C}$ generated by the following directed graph

$$\begin{array}{ccccc} A & \xrightarrow{\gamma} & B & \overset{f}{\underset{g}{\rightrightarrows}} & C \\ & & \downarrow{\scriptstyle \beta} & & \\ & & I & & \end{array}$$

subject to the equation $f \circ \gamma = g \circ \gamma$. Obviously γ is an epimorphism in $\mathcal{C}/I \cong \{0 \to 1\}$ but not in $\mathcal{C}$.

□

The situation for cartesian categories is better due to the following observation:

Proposition 1.3. *Suppose I is an object of category $\mathcal{C}$ in which all binary products $I \times A$ exists for all objects A of $\mathcal{C}$. The forgetful functor* $\mathrm{dom}\colon \mathscr{E}/I \to \mathscr{E}$ *preserves and creates colimits, and therefore it preserves and creates epimorphisms.*

Proof. right adjoint. □

Recall that

Definition 1.4. A category $\mathcal{C}$ is

- **cartesian** if $\mathcal{C}$ has all finite products (including the empty product which is the terminal object).
- **locally cartesian** if $\mathcal{C}$ has a terminal object 1 and moreover, for any object A of $\mathcal{C}$, the slice category $\mathcal{C}/A$ is cartesian.
- **cartesian closed** if $\mathcal{C}$ is cartesian and moreover, $\mathcal{C}$ has all exponential, i.e. for every object A of $\mathcal{C}$, the functor $(-) \times A\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint.
- **locally cartesian closed** if $\mathcal{C}$ has a terminal object 1 and moreover, for any object A of $\mathcal{C}$, the slice category $\mathcal{C}/A$ is cartesian closed.

Proposition 1.5. *1. Any locally cartesian category is cartesian.*

2. Any locally cartesian category has all finite limits.

3. Any locally cartesian closed category is cartesian closed and therefore cartesian as well.

4. Any locally cartesian closed category is locally cartesian, and therefore has all finite limits.

5. The slices of a locally cartesian category are locally cartesian.

6. The slices of a locally cartesian closed category are locally cartesian closed.

Proof. For (1) and (3) use the isomorphism $\mathcal{C}/1 \cong \mathcal{C}$ of categories to transport the involved structure. For (2), notice that products in slices are pullbacks in the underlying category. (4) follows from the fact that every cartesian closed category, by definition, is cartesian. For (5) and (6), notice that for any objectd I of $\mathcal{C}$, the category $\mathcal{C}/I$ has a terminal object, namely $(I, \mathrm{id}\colon I \to I)$, and furthermore, for every object $(X, p\colon X \to I)$ of $\mathcal{C}/I$, we have the isomorphism of categories in below.

$$\begin{aligned} (\mathcal{C}/I)/(X,p) &\cong \mathcal{C}/X \\ ((X',p'), u\colon (X',p') \to (X,p)) &\longmapsto (X', u\colon X' \to X) \\ h\colon ((X'',p''),u') \to ((X',p'),u) &\longmapsto h\colon (X'',u') \to (X',u) \end{aligned} \tag{1}$$

□

From the theorem above we have the following embeddings of 2-categories:

$$\begin{array}{ccc} \mathfrak{Cat}_{\mathrm{lcc}} & \rightarrowtail & \mathfrak{Cat}_{\mathrm{cc}} \\ \downarrow & & \downarrow \\ \mathfrak{Cat}_{\mathrm{lc}} & \rightarrowtail & \mathfrak{Cat}_{\mathrm{c}} \end{array}$$

Here $\mathfrak{Cat}_{\mathrm{lcc}}$ is the 2-category of locally cartesian closed categories, $\mathfrak{Cat}_{\mathrm{cc}}$ is the 2-category of cartesian closed categories, $\mathfrak{Cat}_{\mathrm{lc}}$ is the 2-category of locally cartesian categories, and $\mathfrak{Cat}_{\mathrm{c}}$ is the 2-category of cartesian categories. In each case the morphisms are the functors which preserve the appropriate structures.

Suppose $\mathcal{C}$ is a locally cartesian category and consider the self-indexing pseudo-functor $\mathbb{C} \colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{Cat}$ associated to $\mathcal{C}$: it takes an object I of $\mathcal{C}$ to the slice category $\mathcal{C}/I$ and takes a morphism $f \colon J \to I$ of $\mathcal{C}$ to the functor $\Delta_f \colon \mathcal{C}/I \to \mathcal{C}/J$ where Δ_f is given by the pullback along f (aka reindexing along f). Notice that $\mathbb{C} \colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{Cat}$ factors through $\mathfrak{Cat}_{\mathrm{c}} \rightarrowtail \mathfrak{Cat}$ since each slice is cartesian and moreover, the reindexing preserves finite limits. More is true: $\mathbb{C}$ factors through $\mathfrak{Cat}_{\mathrm{lc}}$ because of part (5) of theorem (1.5).

Remark 1.6. Given a locally cartesian closes category $\mathcal{C}$, the self-indexing pseudo-category $\mathbb{C} \colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{Cat}$ factors through $\mathfrak{Cat}_{\mathrm{lcc}} \rightarrowtail \mathfrak{Cat}$. This is because the reindexing functors preserve the exponentials in the slices. To see this, first note that [♠1:♠]

Example 1.7. The category of small categories is an example of a cartesian closed category which is not locally cartesian closed. To see why, for a natural number m, let $[m]$ be the standard m-simplex considered as a category, i.e. $[m]$ has m objects $0, 1, \ldots, m$ and has exactly one arrow between any two objects (i.e. it is a linear order).

Consider the inclusion $[1] \rightarrowtail [2]$ fixing the endpoints. Now, observe that $p^* \colon \mathcal{C}at/[2] \to \mathcal{C}at/[1]$ can not have a right adjoint since it does not preserve colimits. To see the latter, notice that the following pushout square in $\mathcal{C}at/[2]$

$$\begin{array}{ccc} ([0],\{1\}) & \xrightarrow{\;1\;} & ([1],[0,1]) \\ {\scriptstyle 0}\downarrow & \lrcorner & \downarrow{\scriptstyle [0\leq 1]} \\ ([1],[1,2]) & \xrightarrow[{[1\leq 2]}]{} & ([2],\mathrm{id}) \end{array}$$

once pulled back to $\mathcal{C}at/[1]$ yields the diagram

$$\begin{array}{ccc} ([0],\{1\}) & \xrightarrow{\;1\;} & ([1],[0,1]) \\ {\scriptstyle 0}\downarrow & \lrcorner & \downarrow{\scriptstyle [0\leq 1]} \\ ([1],[1,2]) & \xrightarrow[{[1\leq 2]}]{} & ([2],\mathrm{id}) \end{array}$$

which clearly is not a pushout.

Construction 1.8. Here we show that how to construct

1. the re-indexing from the binary cartesian products in slices,

2. the dependent products from the exponentials in slices.

For (1), define $(\Delta_f(X), f \times p) \triangleq (X, p) \times (J, f)$ in $\mathcal{C}/I$, and let the $\mathcal{C}$-morphisms $\Delta_f(p) \colon f \times p \to f$ and $\varepsilon_p \colon f \times p \to p$ be the product projection. Now, the universal property of $(\Delta_f(X), f \times p)$ says exactly that the diagram below is a pullback square in $\mathcal{C}$.

$$\begin{array}{ccc} \Delta_f(X) & \xrightarrow{\varepsilon_f(p)} & X \\ {\scriptstyle \Delta_f(p)}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle p} \\ J & \xrightarrow[f]{} & I \end{array} \tag{2}$$

Let's take a moment to see $\varepsilon_f(p)$ in terms of maps of families indexed by I: indeed, $\varepsilon_f(p)$ corresponds to the map of families $(\sum_{j:J_i} X_{f(j)} \mid i : I) \equiv (\sum_{j:J_i} X_i \mid i : I) \equiv (J_i \times X_i \mid i : I) \to (X_i \mid i : I)$ which projects a pair (j, x) to its second coordinate x.

According to the table (1), we think of the dependent product $\Pi_f(Y)$ as the family $(\prod_{j:J_i} Y_j \mid i : I)$. Therefore, we can identify an element of this dependent product by an assignment $J_i \to \sum_{j:J_i} Y_j$ whose first coordinate assignment is identity on J_i, i.e. a map $f \to \Sigma_f(q) = f \circ q$ in $\mathcal{C}/I$ whose composite with q is the identity morphism on J.

Therefore, for (2), given the left hand side diagram in $\mathcal{C}$, we obtain $\Pi_f(q)$ in the category $\mathcal{C}/I$, as shown in below on the right hand side.

$$\begin{array}{ccc} Y & & \\ {\scriptstyle q}\big\downarrow & & \\ J & \xrightarrow[f]{} & I \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} (\Pi_f Y, \Pi_f(q)) & \xrightarrow{i_f(q)} & (Y, \Sigma_f q)^{(J,f)} \\ \big\downarrow & \ulcorner & \big\downarrow{\scriptstyle q^f} \\ (I, 1_I) & \xrightarrow[\widehat{id_f}]{} & (J, f)^{(J,f)} \end{array} \quad = \quad \begin{array}{ccc} \left(\prod_{j:J_i} Y_j \mid i : I\right) & \rightarrowtail & J_i^{\left(\sum_{j:J_i} Y_j \mid i:I\right)} \\ \big\downarrow & \ulcorner & \big\downarrow \\ (1 \mid i : I) & \xrightarrow[\widehat{id_f}]{} & (J_i^{J_i} \mid i : I) \end{array} \tag{3}$$

where $\widehat{id_f}$ is the transpose of $(I, 1_I) \times (J, f) \cong (J, f)$ through the adjunction $(-) \times (J, f) \dashv (-)^{(J,f)}$.

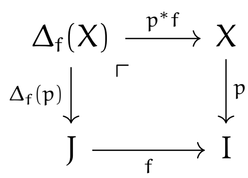
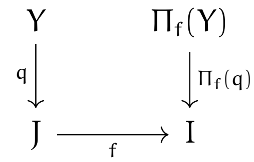
Remark 1.9. It is easy to see that the construction above is functorial w.r.t. q. Alternatively, the functoriality of Π_f can be seen an special case of Proposition in below.

Remark 1.10. For the unique morphism $!_I \colon I \to 1$ to the terminal object, we write Δ_I for Δ_f. We have the natural isomorphism $\Sigma_I \circ \Delta_I \cong (-) \times I$ of functors.

We note that the functor Δ_I is not full, since even between two constant families of spaces there are usually many non-constant I-indexed families of maps: In fact,

$$\mathcal{C}/I(\Delta_I(X), \Delta_I(Y)) \cong \mathcal{C}(X \times I, Y) \cong \mathcal{C}(I, Y^X)$$

The table below provides the internal familial version of the operations Σ, Δ, Π in a locally cartesian closed category.

Internal (familial) logic of $\mathcal{C}$	Category $\mathcal{C}$
$(X_i \mid i : I)$	$(X, p\colon X \to I) \in \mathcal{C}/I$
$(f_i\colon X_i \to Y_i \mid i : I)$	$(X,p) \to (Y,q) \in \mathcal{C}/I$
$(x_i\colon X_i \mid i : I)$	(section) $x\colon (I, 1_I) \to (X,p) \in \mathcal{C}/I$
$\sum_{f\colon J\to I} (Y_j \mid j : J) \triangleq (\sum_{j:J_i} Y_j \mid i : I)$	$\dfrac{(Y, q\colon Y \to J) \in \mathcal{C}/J \qquad (J, f\colon J \to I) \in \mathcal{C}/I}{(Y, f\circ q\colon Y \to I) \in \mathcal{C}/I}$
$\Delta_{(f\colon J\to I)}(X_i \mid i : I) \triangleq (X_{f(j)} \mid j : J)$	
$\prod_{f\colon J\to I} (Y_j \mid j : J) \triangleq (\prod_{j:J_i} Y_j \mid i : I)$	

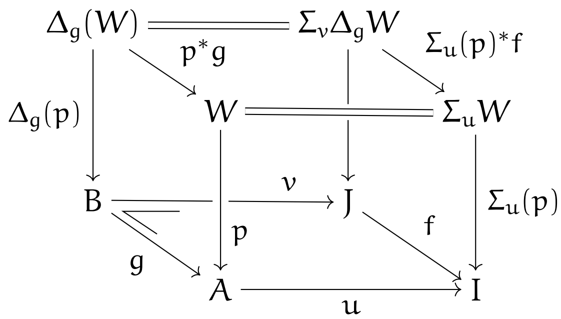
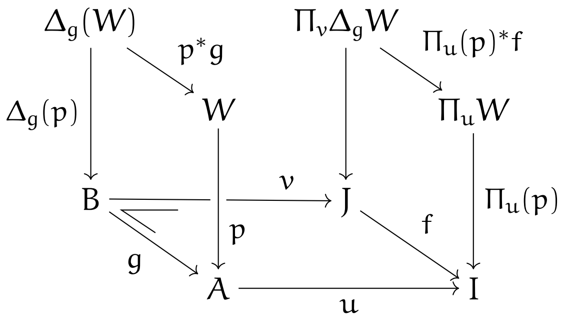
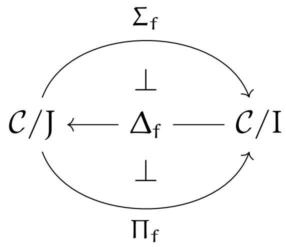
Parametric equations	Commutativities in $\mathcal{C}$
$\dfrac{B \equiv (A_i \times J_i \mid i : I)}{\sum_{a\in A_{f(j)}} W_a \cong \sum_{b\in B(j)} W_{g(b)}}$	
$\dfrac{B \equiv (A_i \times J_i \mid i : I)}{\prod_{a\in A_{f(j)}} W_a \cong \prod_{b\in B(j)} W_{g(b)}}$	

Table 1: Internal logic of locally cartesian closed categories

REMARK 1.11. In particular the operation $\sum_J \triangleq \sum_{!_J\colon J\to 1}$ applied to a family $(Y_j | j : J)$ yields the total space $\sum_{j:J} Y_j \cong Y$. Similarly, for Π_J, we have $\prod_J (Y_j \mid j : J) = \prod_{j:J} Y_j$, the total space of section of q, which is an object of $\mathcal{C}$.

Using Construction (1.8) we arrive at the following adjunction of categories:

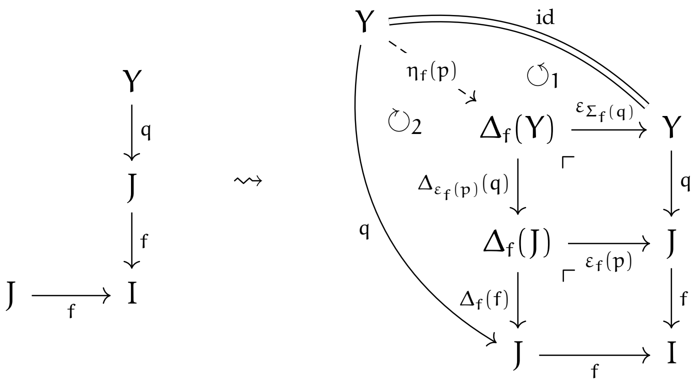
Theorem 1.12. *Given a morphism $f\colon J \to I$ in a locally cartesian closed category, we have the triple adjunction of categories.*



(4)

Proof. The proof is basically a construction of the unit and counit of the adjunctions together with the proofs of their triangle equalities, and a little bit of conceptual meditation.

We start from the adjunction $\Sigma_f \dashv \Delta_f$. In the diagram below, we see the unit $\eta^{(\Sigma)}\colon \mathrm{Id}_{\mathcal{C}/J} \Rightarrow \Delta_f \circ \Sigma_f$ given component-wise by the unique (dashed) morphism to the pullback cone over f, fq



(5)

How do we think of $\eta_f(p)$? The morphism $\Delta_f\Sigma_f(q)\colon \Delta_f(Y) \to J$ corresponds to the family $(\sum_{j':J_{f(j)}} Y_{j'} \mid j : J)$. The morphism $\eta_f(p)$, at the component $j : J$, takes $y : Y_j$ to $(j, y) : \sum_{j':J_{f(j)}} Y_{j'}$.

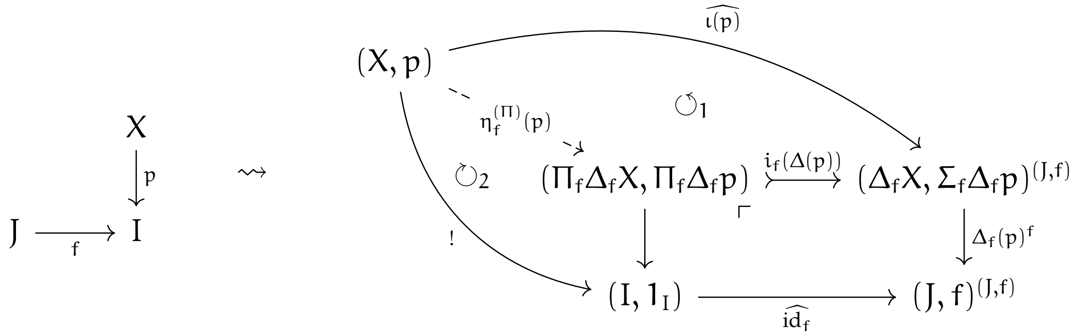
Before moving on to the counit, let us mention here the interesting property of the morphism $\Delta_f(f)\colon \Delta_f(J) \to J$: it corresponds to the family $(J_{f(j)} \mid j : J)$ and it has a generic element (section) $j\colon J \to \Delta_f(J)$ And the counit $\varepsilon\colon \Sigma_f \circ \Delta_f \Rightarrow \mathrm{Id}_{\mathcal{C}/I}$ of this adjunction is given component-wise by the morphism $\varepsilon_{(X,p)}\colon (\Delta_f(X), f \circ \Delta_f(p)) \to (X, p)$ constructed in (1.8) part (1).

The first triangle equality $\varepsilon_{\Sigma_f(q)} \circ \Sigma_f(\eta_q) = \mathrm{id}_{\Sigma_f(q)}$ is witnessed by the commutativity $\circlearrowleft_1$ in the diagram (6). The second triangle equality $\mathrm{id}_{\Delta_f(p)} = \Delta_f(\varepsilon_p) \circ \eta_{\Delta_f(p)}$ follows from the universal property of $\Delta_f(p)$, introduced in the diagram (2), and the commutativity $\circlearrowleft_2$ in the diagram (6).

Next we prove the adjointness $\Delta_f \dashv \Pi_f$. First note that for $p\colon X \to I$ in $\mathcal{C}/I$, $\Pi_f\Delta_f(p)$ corresponds to the family $(X_i^{J_i} \mid i : I)$. Also, as shown in the Construction (1.8), we have $\iota(p)\colon f \times p \cong \Sigma_f\Delta_f p$, and therefore its transpose $\widehat{\iota(p)}$ has the type $p \to (\Sigma_f\Delta_f p)^f$. The corresponding map of families is given by $(X_i \to (J_i \times X_i)^{J_i} \mid i : I)$ given by currying of the identity on $(J_i \times X_i \mid i : I)$.

Obviously we have $\Delta_f(p) \circ \iota(p) = \mathtt{pr}_2\colon (J, f) \times (X, p) \to (J, f)$, which in turn is equivalent to the transposed equation $\Delta_f(p)^f \circ \widehat{\iota(p)} = \widehat{\mathrm{id}_f} \circ !_{(X,p)}$, and therefore, we obtain the unit

$\eta^{(\Pi)}\colon \mathrm{Id}_{\mathcal{C}/I} \to \Pi_f \circ \Sigma_f$, at component $p\colon X \to I$, as the unique morphism (in RHS diagram in below) obtained by the universal property (see diagram (8)) of $\Pi_f \Delta_f(p)$ as follows:

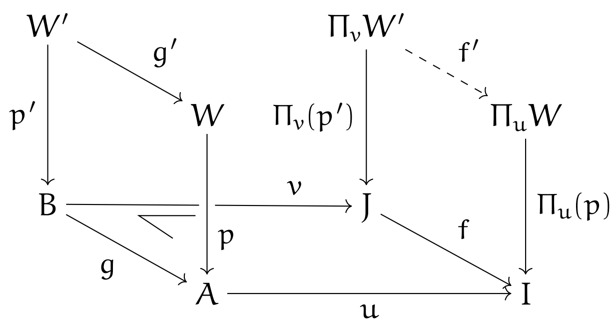


(6)

Moreover, the counit $\varepsilon^{(\Pi)}$ at the component q is given by a morphism ε [♠2:... ♠] Note that the family corresponding to $\Delta_f \Pi_f(q)$ is $(\prod_{j':J_{f(j)}} Y_{j'} \mid j : J)$. Then the morphism is fibrewise given by $\varepsilon^{(\Pi)}(j)\colon \prod_{j':J_{f(j)}} Y_{j'} \to Y_j$ by evaluating at $j : J_{f(j)}$. □

Construction 1.13. The proof above also provides us with a method to get exponentials in slices from the dependent products. Given $(X, p\colon X \to I)$ and $(Y, q\colon Y \to I)$ the exponential $(Y, q)^{(X,p)}$ is given by $(\Pi_p \Delta_p Y, \Pi_p \Delta_p(q))$. Internally, it corresponds to th family $(\prod_{x:X_i} Y_{p(x)} \mid i : I) = (Y_i^{X_i} \mid i : I)$.

Proposition 1.14. *Suppose as in the diagram below, the morphism (square) $(u, v)\colon g \to f$ is a cartesian square, i.e. it is a morphism in the subcategory $\mathscr{E}^{\mathrm{cart}} \hookrightarrow \mathscr{E}^{\to}$. If the vertical square on the left commutes then there is a morphism $f'\colon \Pi_v W' \to \Pi_u W$ which makes the vertical square on the right commute.*

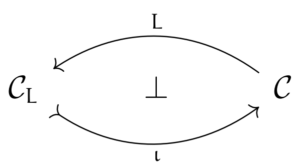


(First Proof) We obtain f' as the transpose of the composite

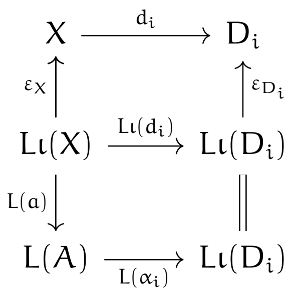
$$\Delta_u \Sigma_f \Pi_v(p') \xrightarrow{\cong} \Sigma_g \Delta_v \Pi_v(p') \xrightarrow{\Sigma_g \varepsilon_v(p')} \Sigma_g p' \xrightarrow{g'} p \,.$$

(Second Proof) Note that the family corresponding to $\Pi_v(p')$ is $(\prod_{b:B_j} W'_b \mid j : J)$ and the family corresponding to $\Pi_u(p)$ is $(\prod_{a:A_i} W_a \mid i : I)$. The morphism f' takes $(j : J, s : \prod_{b:B_j} W'_b)$ to $\big(f(j) : I, \lambda a.\, g'(s\langle j, a\rangle) : \prod_{a:A_{f(j)}} W_a\big)$ where $\langle j, a\rangle$ is the unique morphism to the pullback B which composes with v to j, and with g to a.

Recall that a **reflective localization** can be formulated as an adjoint situation



where the functor ι is an embedding (i.e. full and faithful). Note that just form this setup we immediately get more: L creates limits. To see this, suppose $D\colon J \to \mathcal{C}_L$ is a diagram in $\mathcal{C}_L$ such that ιD has a limit $(A, \alpha_\bullet)$ in $\mathcal{C}$. We claim that $(L(A), \varepsilon_{D_\bullet} \circ L(\alpha_\bullet))$ is the limit of the diagram D in $\mathcal{C}_L$: Take a cone $(X, d_\bullet)$ over D. By functoriality of ι we get the cone $\iota(X), \iota(d_\bullet)$ over ιD and hence this cone must factor through $\alpha_\bullet$ through a unique morphism $a\colon \iota(X) \to A$. Now, observe that all the rectangles in the diagram



commute. Since i is an embedding the counit ε of adjunction is an isomorphism. Hence, the morphism $L(a)\varepsilon_X^{-1}\colon X \to L(A)$ composes with $\varepsilon_{D_i} \circ L(\alpha_i)\colon L(A) \to D_i$ to d_i and is unique with this property(?).

Furthermore, the situation above is called an **exact (reflective) localization** is in addition L preserves finite limits.

Proposition 1.15. *If $\mathcal{C}$ is cartesian closed then so is $\mathcal{C}_L$ and i preserves the exponentials.*

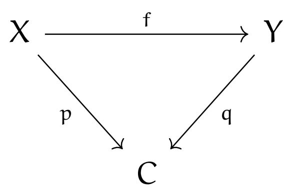
The following example gives more insight into the operations $\Sigma\Delta, \Pi$.

Example 1.16. Suppose $\mathcal{S}et_{fin}$ is the type of finite sets. We define the type of **coloured types** by

$$\sum_{C:\mathrm{Fin}} \sum_{X:\mathcal{U}} X \to C$$

We think of C as the set of 'colours' and a term $(C, p : X \to C)$ as a 'colouring' of type X: the fibre $p^{-1}(c_i)$ is thought of as all elements in X with the colour c_i.

A map



between colourings p and q is a map from X to Y which preserves the colours.

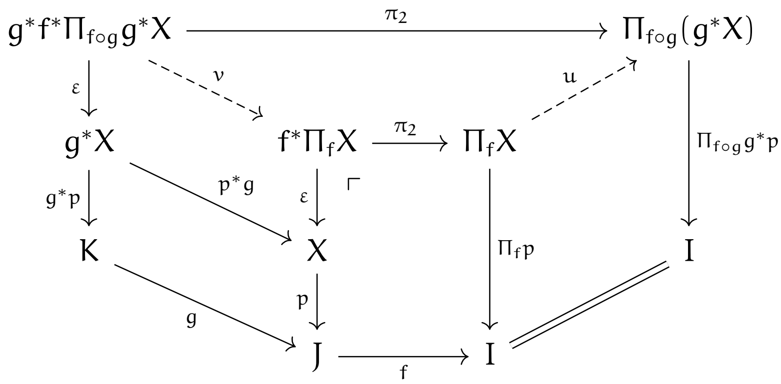
A reindexing of a family $Y \to D$ along a change of colour map $\sigma\colon C \to D$ is a substitution of colours Y by colours in C, i.e. $Y[\sigma]_{c_i} = Y_{\sigma(c_i)}$: All elements coloured by c_i in $Y[\sigma]$ are the ones coloured $\sigma(c_i)$ in Y.

Suppose for a colour $d : D$ the map σ takes exactly the colours $c_1^d, \dots, c_n^d$ to the colour d. Elements of $\Sigma_\sigma X$ coloured by d are all of the elements of X coloured by any of $c_1^d, \dots, c_n^d$. Similarly, the elements of $\Pi_\sigma X$ coloured d are n-tuples drawn from X, each of which coloured by either $c_1^d, \dots, c_n^d$ and they all have different colours.

Remark 1.17. Π is functorial in its arguments , i.e. $\Pi_{f\circ g} = \Pi_f \circ \Pi_g$ where $f \colon K \to J$ and $g \colon J \to I$. Suppose $\zeta \colon Z \to K$ is morphism corresponding to the family $(Z_k \mid k : K)$. Then the family corresponding to $\Pi_f \circ \Pi_g(\zeta)$. is $(\prod_{j:J_i} \prod_{k:K_j} Z_k \mid i : I)$ and the family corresponding to $\Pi_{f\circ g}(\zeta)$. is $\prod_{k:\sum_{j:J_i} K_j} Z_k \mid i : I$. Therefore, we have the isomorphism of families

$$(\prod_{j:J_i} \prod_{k:K_j} Z_k \mid i : I) \cong (\prod_{k:\sum_{j:J_i} K_j} Z_k \mid i : I)$$

Construction 1.18. Take I, J, K, f, g to be the same as in the remark above, and take $p \colon X \to J$. There exist morphisms u, v as in the diagram below such that all the induced subdiagrams commute.



Setting $Y \triangleq g^*X$, in terms of families, the morphism $u \colon \Pi_f X \to \Pi_{f\circ g} Y$ is the map

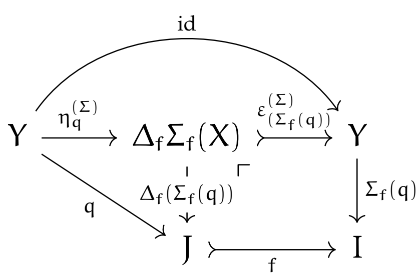
$$(\prod_{j:J(i)} X(j) \mid i : I) \to (\prod_{(j,k):\sum_{j:J(i)} K(j)} Y(k) \mid i : I) \quad , \; x \mapsto \lambda(j,k).\, x(j)$$

The morphism u is obtained by applying Π_f to $\eta_p \colon p \to \Pi_g g^* p$, since by Remark (1.17) $\Pi_{f\circ g}(g^*p) = \Pi_f \Pi_g g^* p$.

Proposition 1.19. *Given a morphism* $f \colon J \to I$ *in a locally cartesian closed category*

- f *is a monomorphism if and only if* $\eta_f^{(\Sigma)}$ *is an isomorphism if and only if* $\varepsilon_f^{(\Pi)}$ *is an isomorphism.*
- *[♠3:???♠]*

Proof. We construct an inverse to $\varepsilon_f^{(\Pi)}$ as follows. First note that because of an adjunction triangle equality we have $\varepsilon^{(\Sigma)}_{(\Sigma_f(q))} \circ \eta_q^{(\Sigma)} = \mathrm{id}_Y$.



(7)

Now, $\eta_q^{(\Sigma)}$ is a section of a monomorphism, and therefore it is an isomorphism. Using adjunctions $\Sigma_f \dashv \Delta_f \dashv \Pi_f$ we have the following correspondences:

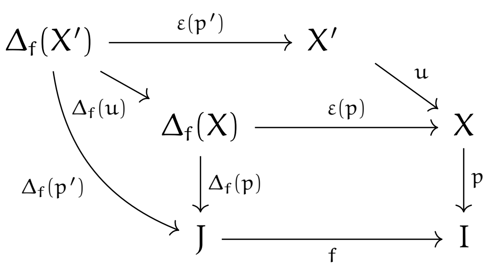
$$\dfrac{\dfrac{\Delta_f\Sigma_f(q) \to q}{\Sigma_f(q) \to \Pi_f(q)}}{q \to \Delta_f\Pi_f(q)}$$

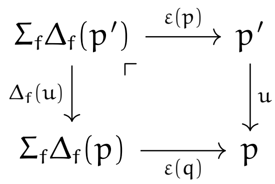
Therefore, we get an inverse for $\varepsilon_f^{(\Pi)} \colon \Delta_f\Pi_f(q) \to q$.

<u>`Alternative proof`</u>: When f is a monomorphism $J_{f(j)} \cong \{j\}$ and therefore, the family $(\prod_{j':J_{f(j)}} Y_{j'} \mid j : J)$ is isomorphic to $(Y_j \mid j : I)$. □

Remark 1.20. The unit and counit of the adjunction $\Sigma_f \dashv \Delta_f$ are cartesian, i.e. the involved naturality squares are pullbacks. We show this for the case of counit and leave it to the readers to figure the cartesianness of the unit for themselves.

For any morphism $u \colon (X', p') \to (X, p)$ in the slice category $\mathcal{C}/I$, the (naturality) commutative square on the right hand side is a pullback square in $\mathcal{C}/I$. This follows from the pasting lemma of pullbacks in the diagram on the left hand side in the category $\mathcal{C}$.

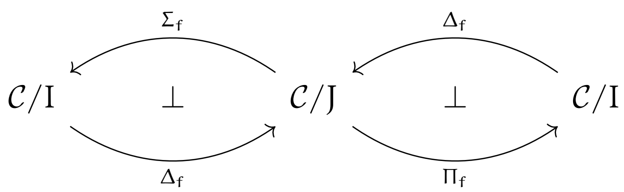




Remark 1.21. The endofunctor

$$-^{(J,f)} \colon \mathcal{C}/I \to \mathcal{C}/I$$

– which computes the exponentials $(X, p)^{(J,f)}$ – is indeed isomorphic to the composite $\Pi_f \circ \Delta_f$. This can be easily seen when we compute in the internal language: the family corresponding to the morphism $\Pi_f \circ \Delta_f(X, p)$ is $\prod_{j:J_i|i:I} \equiv X_i^{J_i}$. Categorically, this is seen by the fact that $\Pi_f \circ \Delta_f$ is the right adjoint to $\Sigma_f \circ \Delta_f = () \times f$.



It is worth mentioning in passing that the evaluation morphism (i.e. counit of the exponentiation) $f \times p^f \to p$ is given by the composite $(J, f) \times (X, p)^{(J,f)} \xrightarrow{i_f(\widehat{\Delta(p)})} (\Sigma_f\Delta_f(X), \Sigma_f\Delta_f(p)) \xrightarrow{\varepsilon^{(\Sigma)}(p)} (X, p)$. Component-wise this composite morphism is simply the evaluation $J_i \times X_i^{J_i} \to X_i$.

Corollary 1.22. *If a category $\mathcal{C}$ is locally cartesian, then any morphism f is exponentiable if and only if the right adjoint Π_f exists. A locally cartesian category $\mathcal{C}$ is locally cartesian closed if and only if for every morphism f of $\mathcal{C}$, the the right adjoint Π_f exists.*

Example 1.23. We list some of the special cases of the adjunction (4):

- When $I = 1$ is the terminal object and $f = !_J$ is the unique morphism $J \to 1$, we have $\Sigma_J(Y, q) \triangleq \Sigma_f(q) = Y$ and $\Delta_J(X, !_X) = \mathrm{pr}_1 \colon J \times X \to J$, and therefore, $\Sigma_J \Delta_J(X) = J \times X$. The unit of adjunction $\Sigma_J \dashv \Delta_J$ is given by the morphism $(q, \mathrm{id}) \colon Y \to J \times Y$ natural in $q \colon Y \to J$. Also, in this special situation, we have a functor $\Pi_J \colon \mathcal{C}/J \to \mathcal{C}$ such that $\Pi_J \Delta_J X \cong X^J$, and $\Pi_J(Y, q \colon Y \to J)$ is the object of sections of q, since the diagram (8) takes a simpler shape.

$$
\begin{array}{ccc}
Y & & \\
{\scriptstyle q}\big\downarrow & & \\
J & \longrightarrow & 1
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
\Pi_J(q) & \overset{i_J(q)}{\rightarrowtail} & Y^J \\
\big\downarrow & \ulcorner & \big\downarrow {\scriptstyle q^J} \\
1 & \underset{\widehat{\mathrm{id}_J}}{\longrightarrow} & J^J
\end{array}
\tag{8}
$$

Note the internal version of the adjunction $\Delta_J \dashv \Pi_J$ is expressed by the rule

$$W \to \prod_{j:J} X_j \cong \prod_{j:J} (W \to X_j)$$

- When $J = 1$ and $f = i_0 \colon 1 \to I$, we get an adjunction

$$
\mathcal{C} \;\underset{\Pi_{i_0}}{\overset{\Sigma_{i_0}}{\underset{\bot}{\overset{\bot}{\leftarrow\!\!-\; \Delta_{i_0} \;-\!\!-}}}}\; \mathcal{C}/I
\tag{9}
$$

where $\Delta_{i_0}(X, p \colon X \to I)$ is isomorphic to the fibre X_{i_0}, the dependent sum $\Sigma_{i_0}(Y)$ is given by the family $(\sum_{\{\_:1 \mid i = i_0\}} Y_{i_0} \mid i : I)$, and finally the dependent product $\Pi_{i_0}(Y)$ is given by the family $(\prod_{\{\_:1 \mid i = i_0\}} Y_{i_0} \mid i : I)$. The exponential $(X, p)^{(1, i_0)}$ then is computed by $\Pi_{i_0} \Delta_{i_0}(X, p)$ which corresponds to the internal family $(\prod_{\{\_:1 \mid i = i_0\}} X_{i_0} \mid i : I)$. The latter is sometimes called skyscraper presheaf/sheaf in the context of sheaf theory.[1]

- When $J \triangleq A \times B$, $I \triangleq A$, and $f \triangleq \mathrm{pr}_1 \colon A \times B \to A$, the objects $\Sigma_{\mathrm{pr}_1}(Y, q)$, $\Delta_{\mathrm{pr}_1}(X, p)$, and $\Pi_{\mathrm{pr}_1}(Y, q)$ correspond to the internal families $(\sum_{(b:B)} Y_{a,b} \mid a : A)$, $(X(a) \mid a : A, b : B)$, and $(\prod_{b:A} Y_{a,b} \mid a : A)$, respectively. Also, the exponential $(A \times B, \mathrm{pr}_1)^{(X,p)}$ corresponds to the internal family $(X(a)^B \mid a : A)$.

Proposition 1.24. *Consider the diagrams*

$$
\begin{array}{ccc}
\Delta_f(X) & \overset{p^*f}{\longrightarrow} & X \\
{\scriptstyle \Delta_f(p)}\big\downarrow & \ulcorner & \big\downarrow {\scriptstyle p} \\
J & \underset{f}{\longrightarrow} & I
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y & & \Pi_f(Y) \\
{\scriptstyle q}\big\downarrow & & \big\downarrow {\scriptstyle \Pi_f(q)} \\
J & \underset{f}{\longrightarrow} & I
\end{array}
$$

- *The sections of* f^*p *are the diagonal fillers of the square on the left.*

[1] and is classically equivalent to $1 + X_{i_0}$.

- *The sections of* $\Pi_f q$ *are exactly the sections of* q.

Proof. From the adjunction $\Sigma_f \dashv \Delta_f$ we have the bijection

$$\mathcal{C}/J\big((J, \mathrm{id}_J), (\Delta_f(X), \Delta_f(p)\big) \cong \mathcal{C}/I\big((J, f), (X, p)\big).$$

Therefore, using this bijection, a section s of $\Delta_f(p)$ corresponds to a morphism $\widehat{s}$ with the property that $p \circ \widehat{s} = f$. One of the triangle identity of the adjunction states that $\varepsilon_p^{(\Sigma)} \circ \Sigma_f(s) = \widehat{s}$ and the latter equation entails that $\widehat{s} \circ \Delta_f(p) = \varepsilon_p^{(\Sigma)} \circ s \circ \Delta_f(p) = p^* f$.

For the second part follows in a rather similar fashion from the local bijection

$$\mathcal{C}/J\big((J, \mathrm{id}_J), (Y, q)\big) \cong \mathcal{C}/I\big((I, \mathrm{id}_I), (\Pi_f Y, \Pi_f q)\big).$$

from the adjunction $\Delta_f \dashv \Pi_f$. Intuitively, for a section t of q, the section $\widehat{t}$ is the family of elements $\widehat{t}(i) : \prod_{j:J_i} Y_j$ defined as $\widehat{t}(i)\, j \triangleq t(j)$. Also $\widehat{s}\, j \triangleq s(fj)j$. □

Remark 1.25 (Beck–Chevalley conditions). Suppose we have a commuting diagram

$$\begin{array}{ccc} B & \xrightarrow{v} & J \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[u]{} & I \end{array}$$

We get natural transformations $\Sigma_g \Delta_v \Rightarrow \Delta_u \Sigma_f$ and $\Sigma_v \Delta_g \Rightarrow \Delta_f \Sigma_u$ component-wise given by the following maps of families:

$$\Big(\sum_{b:B_a} Y_{v(b)} \mid a : A \Big) \to \Big(\sum_{j:J_{u(a)}} Y_j \mid a : A \Big) \;,\; (a, (b, y)) \mapsto (a, (v(b), y)) \tag{10}$$

$$\Big(\sum_{b:B_j} X_{g(b)} \mid j : J \Big) \to \Big(\sum_{a:A_{f(j)}} X_a \mid j : J \Big) \;,\; (j, (b, x)) \mapsto (j, (g(b), x)) \tag{11}$$

Additionally, we get natural transformations $\Delta_u \Pi_f \Rightarrow \Pi_g \Delta_v$ and $\Delta_f \Pi_u \Rightarrow \Pi_v \Delta_g$ component-wise given by the following maps of families:

$$\Big(\prod_{j:J_{u(a)}} Y_j \mid a : A \Big) \to \Big(\prod_{b:B_a} Y_{v(b)} \mid a : A \Big) \;,\; (a, s) \mapsto (a, \lambda b.\, s(v(b))) \tag{12}$$

$$\Big(\prod_{a:A_{f(j)}} X_a \mid j : J \Big) \to \Big(\prod_{b:B_j} X_{g(b)} \mid j : J \Big) \;,\; (j, s) \mapsto (j, \lambda b.\, s(g(b))) \tag{13}$$

The **Beck–Chevalley conditions** says that if in addition the square above is cartesian then all four of the natural transformations above are in fact invertible.

The commutativity of the cube diagrams in the table (1) are two of these isomorphism applied a morphism $p \colon W \to A$, corresponding to the family $(W_a \mid a \in A)$.

$$\begin{array}{ccc} \mathcal{C}/B & \xleftarrow{\Delta_v} & \mathcal{C}/J \\ {\scriptstyle \Sigma_g}\downarrow & \cong & \downarrow{\scriptstyle \Sigma_f} \\ \mathcal{C}/A & \xleftarrow[\Delta_u]{} & \mathcal{C}/I \end{array} \;,\quad \begin{array}{ccc} \mathcal{C}/B & \xrightarrow{\Sigma_v} & \mathcal{C}/J \\ {\scriptstyle \Delta_g}\uparrow & \cong & \uparrow{\scriptstyle \Delta_f} \\ \mathcal{C}/A & \xrightarrow[\Sigma_u]{} & \mathcal{C}/I \end{array} \;,\quad \begin{array}{ccc} \mathcal{C}/B & \xleftarrow{\Delta_v} & \mathcal{C}/J \\ {\scriptstyle \Pi_g}\downarrow & \cong & \downarrow{\scriptstyle \Pi_f} \\ \mathcal{C}/A & \xleftarrow[\Delta_u]{} & \mathcal{C}/I \end{array} \;,\quad \begin{array}{ccc} \mathcal{C}/B & \xrightarrow{\Pi_v} & \mathcal{C}/J \\ {\scriptstyle \Delta_g}\uparrow & \cong & \uparrow{\scriptstyle \Delta_f} \\ \mathcal{C}/A & \xrightarrow[\Pi_u]{} & \mathcal{C}/I \end{array} \tag{14}$$

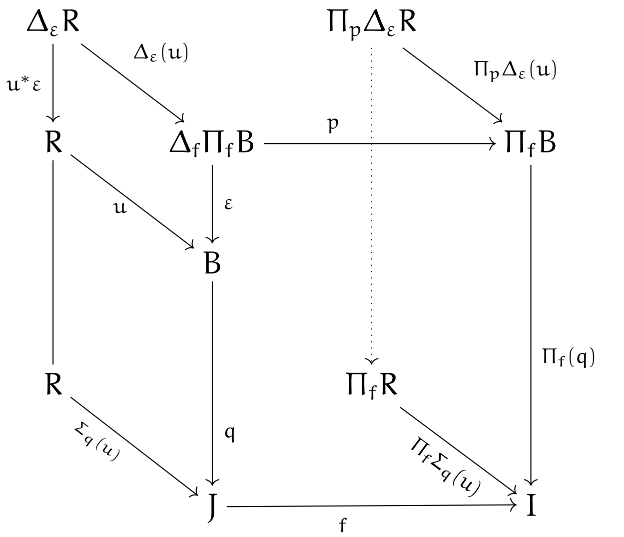
An immediate corollary of the second distributivity law in above is that

Corollary 1.26. *For a morphism* $f\colon J \to I$, *the functor* Δ_f *preserves the exponentials, i.e.* $\Delta_f(q^p) \cong \Delta_f(q)^{\Delta_f(p)}$

Remark 1.27. There is also a distributivity law which internalizes axiom of choice in locally cartesian closed categories. Suppose we have morphisms $u\colon R \to B$, $q\colon B \to J$, and $f\colon J \to I$, in other words, $(R_b \mid b : B)$ is family indexed by B, and B itself is indexed by J, and J by I. We get a canonical map

$$\sum_{s : \prod_{j:J_i} B_j} \prod_{j:J_i} R(j, s(j)) \to \prod_{j:J_i} \sum_{b:B_j} R(j, b) \tag{15}$$

parametrically in $i : I$. The axiom of choice states that this map is an isomorphism. Categorically, the expression on the left of the isomorphism above is given by $\Sigma_{\Pi_f(q)}\Pi_p\Delta_\varepsilon(u)$, and the one on the right is given by $\Pi_f\Sigma_q(u)$, where p is the counit of adjunction $\Sigma_f \dashv \Pi_f$ at the component $\Pi_f(q)$, and ε is the counit of $\Delta_f \dashv \Pi_f$ at the component q. The dotted arrow in the diagram is produced by Proposition (1.14). The axiom of choice states that it is an isomorphism.



(16)

There are three further simplifications to the isomorphism (**??**).

- For I the terminal object 1, we have

$$\prod_{j:J} \sum_{b:B_j} R(j, b) \cong \sum_{s : \prod_{j:J} B_j} \prod_{j:J} R(j, s(j)) \tag{17}$$

- or I the terminal object, $B \triangleq J \times A$, and $q\colon B \to J$ the product projection map, we have

$$\prod_{j:J} \sum_{a:A} R(j, a) \cong \sum_{a:J\to A} \prod_{j:J} R(j, s(j)) \tag{18}$$

  If we think of R as a (generalized) relation on J and A, then the isomorphism above is the standard form of AC in the logic textbooks.

- For $B \triangleq J \times A$, $R \triangleq J \times P$ for some objects A and P, and $u \triangleq J \times v$ for some map $v\colon P \to A$, then we have

$$J \to \Big(\sum_{a:A} P(a)\Big) \cong \sum_{s:J\to A} \prod_{j:J} P(s(j)) \tag{19}$$

Remark 1.28. If we think of isomorphisms as categorification of equations, then general 2-morphisms categorify inequalities. In this light, Beck–Chevalley conditions have a generalization in which the premise stating that the left hand side diagram in (14) is a pullback can be weakened to the premise that the same diagram commutes. In that situation, we have a

1.1 The enrichment aspect

If $\mathcal{C}$ is a locally cartesian closed category we can see the slice category $\mathcal{C}/I$, for every object I, to be enriched over the cartesian monoidal category $\mathcal{C}$. For (X, p) and (Y, q) in $\mathcal{C}/I$, we define the enriched hom as $\underline{\mathrm{Hom}}(p, q) \triangleq \Pi_I \Pi_p \Delta_p(q)$, or in terms of families it is given $\prod_{i:I} Y_i^{X_i}$. The map $q^p \times r^q \to r^p$ corresponds to the enriched composition maps

$$\underline{\mathrm{Hom}}(p, q) \times \underline{\mathrm{Hom}}(q, r) \to \underline{\mathrm{Hom}}(p, r).$$

Given A in $\mathcal{C}$ and $p\colon X \to I$ in $\mathcal{C}/I$ the tensor (copower) $A \otimes p$ is given by $\Sigma_p \Delta_p \Delta_I(A)$ equivalent to the family $(A \times X_i \mid i : I)$.

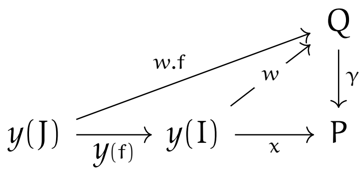
1.2 Locally cartesian closed structures on the presheaf categories

For the discussion in this section we heavily rely on the following lemma. It states a general fact that the slice category of a presheaf category (resp. a diagram category) is again a a presheaf category (resp. a diagram category).

Lemma 1.29. *Let $\mathcal{C}$ be a category and P is a presheaf on $\mathcal{C}$. We have an equivalence of categories:*

$$[\mathcal{C}^{op}, \mathcal{S}et]/P \simeq [(P \rtimes \mathcal{C})^{op}, \mathcal{S}et] \tag{20}$$

Proof. From left to right we take $\gamma\colon Q \to P$ to a presheaf $\widehat{Q}_\gamma$ on the category of elements of P which acts on objects by taking (I, x) to the set of lifts $w\colon y(I) \to Q$ of x against γ. The strict functoriality of this action is given by precomposition, seen in the rightmost diagram in below.



From right to left we take a presheaf $A\colon (P \rtimes \mathcal{C})^{op} \to \mathcal{S}et$ to the presheaf $P.A$ on $\mathcal{C}$ defined as follows:

$$P.A(I) = \coprod_{x \in P(I)} A(I, x)$$

Given a morphism $f\colon J \to I$ in $\mathcal{C}$, we have $P.A(f)\colon P.A(J) \to P.A(I)$ defined by the assignment $\mathsf{in}_x(a) \mapsto \mathsf{in}_{x.f}(a.f)$. The functoriality of $P.A$ follows directly from the functoriality of P and A.

The family of the first projections maps $\mathtt{pr}_1(I)\colon P.A(I) \to P(I)$ – sending (x, a) to x – is natural in I and therefore constitute an object of category $[\mathcal{C}^{op}, \mathcal{S}et]/P$.

It is straightforward to show that these two assignments are functorial. Moreover they are quasi-inverse of each other, that is we have isomorphisms

$$\begin{array}{ccc} P.\widehat{Q}_\gamma & \xrightarrow[\cong]{\alpha} & Q \\ & {\scriptstyle \mathtt{pr}_1}\searrow \quad \swarrow{\scriptstyle \gamma} & \\ & P & \end{array} \qquad \widehat{(P.A)}_{\mathtt{pr}_1} \xrightarrow[\cong]{\beta} A$$

α and β are given componentwise as follows:

$$\alpha_I : P.\widehat{Q}_\gamma(I) = \coprod_{x \in P(I)} \widehat{Q}_\gamma(I, x) = \coprod_{x \in P(I)} \{w \mid \gamma_I(w) = x\} \cong Q(I)$$

$$\beta_I : \widehat{(P.A)}_{\mathtt{pr}_1}(I, x) = \{(x, a) \colon y(I) \to P.A\} \cong A(I, x)$$

natural in I and (I, x), resp. □

The following remark brings into view the fibrational aspect of the equivalence above.

Remark 1.30. A presheaf map $\gamma\colon Q \to P$ in $\mathcal{P}\mathrm{Shv}(\mathcal{C})$ corresponds to a map of discrete fibrations over $\mathcal{C}$, via Grothendieck construction.

$$\begin{array}{ccc} Q \rtimes \mathcal{C} & \xrightarrow{\gamma \rtimes \mathcal{C}} & P \rtimes \mathcal{C} \\ & {\scriptstyle \pi_Q}\searrow \quad \swarrow{\scriptstyle \pi_P} & \\ & \mathcal{C} & \end{array}$$

By the left closure property of discrete fibrations under composition, it follows that $\gamma \rtimes \mathcal{C}$ must be a discrete fibration and therefore, it corresponds to an object in $\mathcal{P}\mathrm{Shv}(X \rtimes \mathcal{C})$.

Remark 1.31. An special case of the lemma above applied to a representable presheaf $P = y(I)$ yields an equivalence

$$[\mathcal{C}^{op}, \mathcal{S}et]/y(I) \simeq [(\mathcal{C}/I)^{op}, \mathcal{S}et] \tag{21}$$

where for a map $\gamma\colon Q \to y(I)$ of presheaves, we have for every $(J, f\colon J \to I)$

$$\begin{array}{ccc} \widehat{Q}_\gamma(J, f) & \rightarrowtail & Q(J) \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \gamma_J} \\ 1 & \xrightarrow[\ulcorner f \urcorner]{} & \mathcal{C}(J, I) \end{array} \tag{22}$$

Under this equivalence the projection $y(I) \times \Delta \to y(I)$ correspond to the presheaf Δ restricted along $\mathrm{dom}^{op}\colon (\mathcal{C}/I)^{op} \to \mathcal{C}^{op}$, that is the presheaf $\Delta|_{\mathcal{C}/I} = \Delta \circ \mathrm{dom}^{op}$.

Proposition 1.32. *Given a morphism $\gamma\colon Q \to P$ of presheaves on $\mathcal{C}$, we have*

$$\mathcal{P}Shv(P \rtimes \mathcal{C})/\widehat{Q}_\gamma \simeq \mathcal{P}Shv(Q \rtimes \mathcal{C}) \tag{23}$$

Proof. Use equivalence (20) in combination with isomorphism (1). □

Example 1.33. The categorical equivalence (20) has some remarkable specializations:

- When $\mathcal{C}$ is a discrete category X (i.e. a set X) we get the simple equivalence $\mathcal{S}\mathrm{et}/X \simeq [X^d, \mathcal{S}\mathrm{et}]$
- When $\mathcal{C}$ is a poset [♠4:♠]
- When $\mathcal{C}$ is the one-object category ΣG, for a group G, and $\mathbb{X}$ a right G-set, then we have

$$\mathrm{BG}/\mathbb{X} \simeq \mathcal{P}\mathrm{Shv}(\mathbb{X} \rtimes G)$$

  where $\mathbb{X} \rtimes G$ is the familiar action groupoid.

Lemma 1.34. *Any presheaf category is cartesian closed.*

Proof. Suppose Γ and Δ are presheaves over $\mathcal{C}$. The exponential presheaf Γ^Δ is given componentwise by

$$\Gamma^\Delta(I) = \mathrm{Hom}_{\mathcal{P}\mathrm{Shv}(\mathcal{C})}(y(I) \times \Delta, \Gamma)$$

The restriction map $\Gamma^\Delta(f)$ of a morphism $f\colon J \to I$ in $\mathcal{C}$ is given by the assignment $\omega \mapsto \omega \circ (y(f) \times \Delta)$. Obviously this assignment is functorial. [♠5:Complete the proof by showing that Γ^Δ is indeed a functor and satisfies the UP for exponential.♠] □

Remark 1.35. More explicitly, an element ω of $\Gamma^\Delta(I)$ is a family $(\omega_J\colon \mathcal{C}(J, I) \times \Delta(J) \to \Gamma(J) \mid J \in \mathrm{Ob}(\mathcal{C}))$ which is natural in J, i.e. $\omega_J(f, \zeta).g = \omega'_J(f \circ g, \zeta.g)$ where on the left side of the equation g acts according to Γ and on the right side g acts according to Δ. This can be further simplified by setting

Remark 1.36. If Δ is a constant presheaf then $\Gamma^\Delta(I) \cong \Gamma(I)^\Delta$.

Proposition 1.37. *Suppose Δ and Γ are presheaves on category $\mathcal{C}$. We have*

$$\Gamma^\Delta(I) \cong \mathrm{Hom}_{[(\mathcal{C}/I)^{op}, \mathcal{S}et]}(\Delta|_{\mathcal{C}/I}, \Gamma|_{\mathcal{C}/I})$$

Proof.

$$\Gamma^\Delta(I) = \mathrm{Hom}_{[\mathcal{C}^{op}, \mathcal{S}\mathrm{et}]}(y(I) \times \Delta, \Gamma) \cong \mathrm{Hom}_{[\mathcal{C}^{op}, \mathcal{S}\mathrm{et}]/y(I)}(y(I) \times \Delta, y(I) \times \Gamma) \cong \mathrm{Hom}_{[(\mathcal{C}/I)^{op}, \mathcal{S}\mathrm{et}]}(\Delta|_{\mathcal{C}/I}, \Gamma|_{\mathcal{C}/I})$$

where the last isomorphism follows from Remark (1.31). □

Proposition 1.38. *For presheaves $P, Q\colon \mathcal{C}^{op} \to \mathcal{S}et$, we have $\mathcal{P}Shv(Q \rtimes \mathcal{C})(1, P \circ \pi_Q) \cong \mathcal{P}Shv(\mathcal{C})(Q, P)$*

Proof. [♠6:write a proof using the foregone facts instead of a direct proof.♠]

(First proof). First set Q to be a representable presheaf yc.

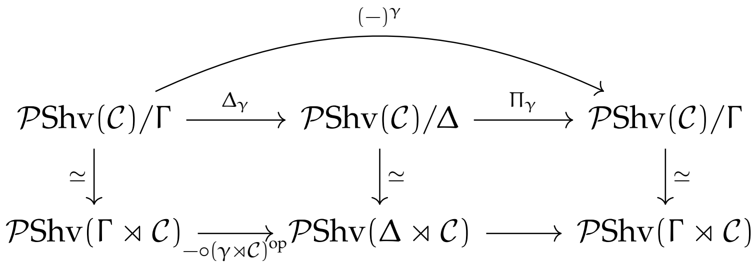
$$\mathcal{P}\mathrm{Shv}(\mathcal{C})(Q, P) \cong \mathcal{P}\mathrm{Shv}(\mathcal{C})(yc, Q^1) \cong \mathcal{P}\mathrm{Shv}(\mathcal{C}/c)(1, Q|_{\mathcal{C}/c}) \cong \mathcal{P}\mathrm{Shv}(Q \rtimes \mathcal{C})(1, P \circ \pi_Q)$$

□

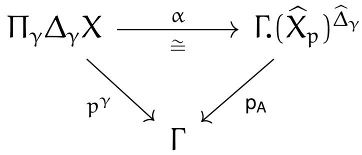
Corollary 1.39. *An immediate consequence of Lemma (1.29) and Lemma (1.34) is that any presheaf category is locally cartesian closed.*

We still would like to have a explicit formula for the exponentials in the slices of presheaf categories. For this our strategy would be to use the equivalence (23) of categories.

Suppose $p\colon X \to A$ and $\gamma\colon \Delta \to \Gamma$ are morphisms in $\mathcal{P}\mathrm{Shv}(\mathcal{C})$. We want to have a closed formula for the elements of presheaf p^γ. For this, we compose the equivalences on the side and bent arrow at the bottom row of the diagram in below.



We obtain the isomorphism



Therefore,

$$\Pi_\gamma\Delta_\gamma X \cong \coprod_{\rho\in\Gamma(I)} \widehat{X_p}^{\widehat{\Delta}_\gamma}(I,\rho) \cong \coprod_{\rho\in\Gamma(I)} \widehat{\Gamma\rtimes\mathcal{C}}(y(I,\rho)\times\widehat{\Delta_\gamma},\widehat{X_p}) \tag{24}$$

Therefore, an element ω of $\Pi_\gamma\Delta_\gamma X(I)$ is a pair (ρ,ω), where $\rho\in\Gamma(I)$ and ω is family

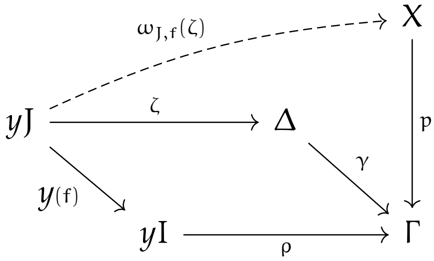
$$\Big(\omega_{J,\nu,f}(\zeta)\in X(J)(\nu) \mid J\in \mathrm{Ob}(\mathcal{C}), \nu\in\Gamma(J), \zeta\in\Delta(J)(\nu), f\colon J\to I.\ \Gamma(f)(\rho)=\nu\Big) \tag{25}$$

which is natural in J, f.

The presentation of ω by the family above can be simplified by removing the dependency on ν: as such ω is equivalent to the family

$$\Big(\omega_{J,f}(\zeta)\in X(J)(\rho.f) \mid J\in \mathrm{Ob}(\mathcal{C}), f\colon J\to I, \zeta\in\Delta(J)(\rho.f)\Big)$$

Expressed only with the diagrams in $\mathcal{P}\mathrm{Shv}(\mathcal{C})$, the data of ω is equivalent to specifying for every f, ζ which makes the floor square commute, we have a (dashed) arrow $\omega_{J,f}(\zeta)$ which makes the outer convex trapezoid commute, i.e. $p\circ\omega_{J,f}(\zeta)=\rho\circ y(f)$.



Moreover, ω satisfies $\omega_{J,f}(\zeta).g=\omega_{J',fg}(\zeta.g)$ for any morphism $g\colon J'\to J$ in $\mathcal{C}$.

Remark 1.40. As a sanity check, if $\Gamma = 1$ is the terminal presheaf then an element of $\Pi_\gamma \Delta_\gamma X = X^\Delta$ is simply given by a family

$$\left(\omega_{J,f}(\zeta) \in X(J) \mid J \in \mathrm{Ob}(\mathcal{C}), f\colon J \to I, \zeta \in \Delta(J) \right)$$

Therefore, this situation matches the exponentials in $\mathcal{P}\mathrm{Shv}(\mathcal{C})$ discussed in Lemma (1.34).

Remark 1.41. We make a remark here about the dependent sums and products in the slice categories of the presheaf categories. Suppose Γ is a presheaf on $\mathcal{C}$ and δ is a morphism

$$\begin{array}{ccc} \Theta & \xrightarrow{\ \delta\ } & \Delta \\ & {\scriptstyle\gamma'}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & \Gamma & \end{array}$$

in $\mathcal{P}\mathrm{Shv}(\mathcal{C})/\Gamma$. Since the diagram

$$\begin{array}{ccc} (\mathcal{P}\mathrm{Shv}(\mathcal{C})/\Gamma)/\gamma' & \xleftarrow{\ \delta_\Gamma^*\ } & (\mathcal{P}\mathrm{Shv}(\mathcal{C})/\Gamma)/\gamma \\ \simeq\downarrow & & \downarrow\simeq \\ \mathcal{P}\mathrm{Shv}(\mathcal{C})/\Theta & \xleftarrow[\ \delta^*\ ]{} & \mathcal{P}\mathrm{Shv}(\mathcal{C})/\Gamma/\Delta \end{array}$$

commutes we have $\Sigma_{\Gamma,\delta} \cong \Sigma_\delta$ and $\Pi_{\Gamma,\delta} \cong \Pi_\delta$