__Def.__ Suppose $\mathcal{K}$ is a 2-category, and $A \xrightarrow{f} C$, $B \xrightarrow{g} C$ are 1-cells in $\mathcal{K}$. A __pseudo-pullback__ of f and g, if it exists, is an object $P \in \mathcal{K}$ together with an equivalence of categories

$$\varphi : \mathcal{K}(X,P) \simeq \mathcal{K}(X,f)/_{\cong}\, \mathcal{K}(X,g)$$

$$\begin{tikzcd}
\mathcal{K}(X,f)/_{\cong}\, \mathcal{K}(X,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_{ps}" very near start] & \mathcal{K}(X,B) \arrow[d, "\mathcal{K}(X,g)"] \\
\mathcal{K}(X,A) \arrow[r, "\mathcal{K}(X,f)"'] & \mathcal{K}(X,C)
\end{tikzcd}$$

Note 7

Suppose E, and B are categories

and $P: E \to B$ is a functor.

P is called isofibration whenever

every isomorphism $f: b' \xrightarrow{\cong} pe$ has

an iso cartesian lift $\tilde{f}: e' \xrightarrow{\cong} e$.

Remark We can think of E, B as

0-cells in CAT and P as 1-cell in

CAT. Then the above definition is

equivalent to say every invertible

2-cell f

$$\begin{tikzcd}
 & 1 \arrow[dl, "e"'] \arrow[dr, "b'"] \arrow[d, phantom, "\overset{f}{\cong}"] & \\
E \arrow[rr, "p"'] & & B
\end{tikzcd}$$

can be lifted to

$$\begin{tikzcd}
 & 1 \arrow[dl, bend right, "e"'] \arrow[dl, bend left, "e'"] \arrow[dr, "b'"] & \\
E \arrow[rr, "p"'] & & B
\end{tikzcd}$$

with $\tilde{f} : e \cong e'$,

that is $\quad p(\tilde{f}) = p * \tilde{f} = f$.

This motivates us to define isofibrations in an arbitrary 2-category (or bicategory) $\mathcal{K}$ as follows:

<u>Def.</u> A 1-cell $E \xrightarrow{p} B$ in $\mathcal{K}$ is an isofibration whenever each invertible 2-cell

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\varphi}{\cong}"] & \\
E \arrow[rr, "p"'] & {} & B
\end{tikzcd}$$

is equal to

$$\begin{tikzcd}
& X \arrow[dl, bend right, "f"'] \arrow[dl, bend left, "\tilde{f}"] \arrow[dl, phantom, "\overset{\tilde{\varphi}}{\cong}"] \arrow[dr, "g"] & \\
E \arrow[rr, "p"'] & & B
\end{tikzcd}$$

for some invertible 2-cell $\tilde{\varphi}$.

Thm. $E \xrightarrow{P} B$ is an isofibration in a

2-category $\mathcal{K}$ iff

$\mathcal{K}(X,E) \xrightarrow{\mathcal{K}(X,P)} \mathcal{K}(X,B)$ is an

categorical isofibration.

Now, we are going to investigate

the pseudo-pullback of 1-cells

along isofibrations in $\mathcal{K}$, and

show that if the pullback exists

in $\mathcal{K}$ it is equivalent to

the pseudo-pullback.

Thm. Suppose

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is given in a 2-category $\mathcal{K}$.

And suppose

$$\begin{tikzcd} H \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\, ps"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is a pseudo-pullback, whereas

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is a strict pullback.

If g is an isofibration, then

$P \simeq H$.

So $\varphi : H \simeq k(X,f)/_{\simeq}\, k(X,g)$

and $\psi : P \simeq k(X,A) \times_{k(X,C)} k(X,B)$

We need to show

$$k(X,f)/_{\simeq}\, k(X,g) \simeq k(X,A) \times_{k(X,C)} k(X,B)$$

$$\begin{tikzcd}
k(X,A) \times_{k(X,C)} k(X,B) \arrow[dr, dashed, "n"] \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_0"'] & & \\
& k(X,f)/_{\simeq}\, k(X,g) \arrow[r] \arrow[d] & k(X,B) \arrow[d, "k(X,g)"'] \\
& k(X,A) \arrow[r, "k(X,f)"'] & k(X,C)
\end{tikzcd}$$

$X \vdash a : A \quad , \quad X \vdash b : B$

Define $\quad n(\langle X \xrightarrow{a} A,\ X \xrightarrow{b} B \rangle) :=$

$$\langle X \xrightarrow{a} A,\ id_{fa=gb},\ X \xrightarrow{b} B \rangle .$$

We show n is essiantially surjective and since n is obviously faithful, then n must be an equivalence.

Take an object $\langle X \xrightarrow{a} A, \begin{tikzcd} X \arrow[r, bend left, "fa"] \arrow[r, bend right, "gb"'] \arrow[r, phantom, "\phi \Downarrow \cong"] & C \end{tikzcd}, X \xrightarrow{b} B \rangle$

$$\begin{tikzcd} & X \arrow[dl, "b"'] \arrow[dr, "fa"] \arrow[d, phantom, "\phi \cong"] & \\ B \arrow[rr, "g"'] & {} & C \end{tikzcd}$$

Can be lifted to

$$\begin{tikzcd} & X \arrow[dl, bend right, "b"'] \arrow[dl, bend left, "b'"] \arrow[dl, phantom, "\phi \cong"] \arrow[dr, "fa"] \arrow[d, phantom, "="] & \\ B \arrow[rr, "g"'] & {} & C \end{tikzcd}$$

Then $\quad n \langle X \xrightarrow{a} A, \; X \xrightarrow{b'} B \rangle$

is isomorphic to $\quad \langle X \xrightarrow{a} A, \; X \overset{\Phi}{\rightrightarrows} C, \; X \xrightarrow{b} B)$

$\square$

Suppose $\mathcal{X}, \mathcal{C}$ are bicategories

and $P : \mathcal{X} \to \mathcal{C}$ is a pseudo-functor

Suppose c is a 0-cell (aka object) in $\mathcal{C}$.

Q1. Under what conditions does strict

pullback

$$\begin{tikzcd}
\mathcal{X}_c^{(s)} \arrow[r] \arrow[d, "P_c"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "P"] \\
1 \arrow[r, "c"'] & \mathcal{C}
\end{tikzcd}$$

exist?

Q2. If $\mathcal{X}_c$ exist is it necessarily

biequivalent to bipullback

$\mathcal{X}_c$? What are the conditions?

$$\begin{array}{c} \mathcal{X} \\ {\scriptstyle P}\downarrow \\ \mathcal{C} \end{array}$$

Define

$\left(\mathcal{X}_c^{(s)}\right)_0 :=$ 0-cells of $\mathcal{X}$ lying over c

that is

$x \in \mathcal{X}_0$ w/ $P(x) = c$

$\left(\mathcal{X}_c^{(s)}\right)_1 :=$ 1-cells of $\mathcal{X}$ lying over

$c \xrightarrow{1_c} c$

that is

$f \in \mathcal{X}_1$ w/ $P(f) = 1_c$

$\left(\mathcal{X}_c^{(s)}\right)_2 :=$ 2-cells of $\mathcal{X}$ between 1-cells of $\mathcal{X}_c^{(s)}$.

Unit in $\mathcal{X}_c^{(s)}$:

$$
\begin{tikzcd}
x \arrow[r, bend left, dashed, "\tilde{1}_c"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\cong"] \arrow[d] & x \arrow[d] \\
c \arrow[r, bend left, "1_c"] \arrow[r, bend right, "P(1_x)"'] \arrow[r, phantom, "\cong"] & c
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{X} \arrow[d, "P"'] \\
\mathcal{C}
\end{tikzcd}
$$

$\bar{u}_{\mathcal{X}} : 1_c \overset{\cong}{\Rightarrow} P(1_x)$

By isofibration property of $P_{x,x}$

$P_{x,x} : \mathcal{X}(x,x) \longrightarrow \mathcal{C}(c,c)$

we can lift $\bar{u}_c$ to

$\tilde{\bar{u}}_{\mathcal{X}} : \tilde{1}_c \overset{\cong}{\Rightarrow} 1_x$

For every x, define $\tilde{1}_c$ to be

the unit morphism (aka 1-cell) at x in

$\mathcal{X}_c^{(s)}$.

Composition :

$$
\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend left=50, dashed, "g \odot f"] \arrow[rr, bend right=50, "g \circ f"'] & Y \arrow[r, "g"'] \arrow[phantom, u, "\cong"] & Z
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "1 = P(f)"'] \arrow[rr, bend left=50, "1"] \arrow[rr, bend right=50, "P(g \circ f)"'] & C \arrow[r, "1 = P(g)"'] \arrow[phantom, u, "\cong \gamma_1"] \arrow[phantom, d, "\cong \pi_{f;g}"] & \mathbb{C}
\end{tikzcd}
$$

γ_1 : from Coherence law of unit in $\mathcal{C}$.

$g \odot f$ is defined to be the composition of f and g in $\mathcal{X}_c^{(s)}$.

Prop. $\mathfrak{X}_c^{(s)}$ is a bicategory.

Proof:

We prove w.associativity of composition and unit; i.e.

in $\mathfrak{X}_c^{(s)}$: (i) $h\odot(g\odot f)\cong(h\odot g)\odot f$

and

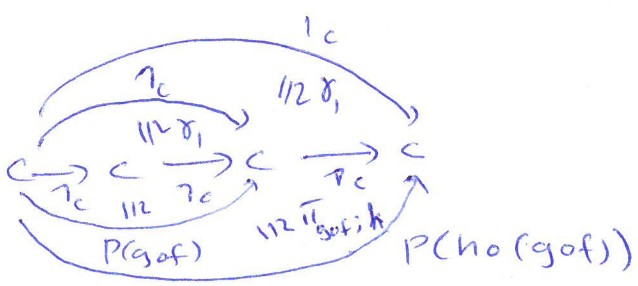
(ii) $f\odot\tilde{1}_c\cong f$

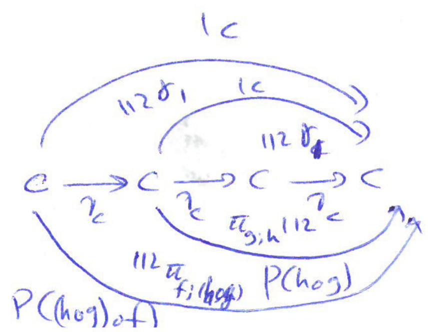
(iii) $\tilde{1}_c\odot g\cong g$ + ...

(i) Suppose 1-cells f, g, h are given

in $\mathfrak{X}_c^{(s)}$.

$$X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{h}W$$





$$
\begin{tikzcd}
\mathfrak{X}(x,w) \arrow[dd, "P_{x,w}"] \\
\\
\mathcal{C}(c,c)
\end{tikzcd}
\qquad
\begin{tikzcd}
h\circledcirc(g\circledcirc f) \arrow[rrr, dashed] \arrow[dr, "\cong"'] & & & (h\circledcirc g)\circledcirc f \arrow[dr, "\cong"] & & \\
& h\odot(g\circ f) \arrow[dr, "\cong"'] & & & (h\circ g)\odot f \arrow[dr, "\cong"] & \\
& & h\circ(g\circ f) \arrow[rrr, "\cong"'] \arrow[dd, mapsto] & & & (h\circ g)\circ f \arrow[dd, mapsto] \\
& 1_c \arrow[rrr, equal] \arrow[dr, "\cong"'] & & & 1_c \arrow[dr, "\cong"] & \\
& & P(h\circ(g\circ f)) \arrow[rrr, "\cong"] & & & P((h\circ g)\circ f)
\end{tikzcd}
$$

Indeed the 2-cell

$h\circledcirc(g\circledcirc f) \xrightarrow{\cong} (h\circledcirc g)\circledcirc f$ is

a unique vertical 2-cell

which makes the rectangle

in the upper layer commutative.

(obtained from uniqueness of

cartesian lift w.r.t. $P_{x,w}$)

(ii) Note that

$$f \circ \tilde{1}_c \cong f \circ \tilde{1}_c .$$

Also

$$f \circ \tilde{1}_c \overset{\tilde{u}_x}{\cong} f \circ 1_x \overset{\theta_f}{\cong} f$$

where θ_f is part of data of bi-category $\mathfrak{X}$ and

$$\tilde{u}_x : 1_x \overset{\cong}{\Rightarrow} \tilde{1}_c .$$

So $f \circ \tilde{1}_c \cong f$. Similarly

$\tilde{1}_c \circ g \cong g$ and there iso 2-cells are coherent with canonical iso 2-cells of weak associativity.