

Street for pedestrians

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Abstract

The singular aim of these notes is to give a less condensed and more clear explanation of Ross Street's significant paper (Street 1974). We highly encourage reader to also refer to (Kock 1995) and (Lack 2000).

1 Pseudo algebras for strict 2-monads

DEFINITION 1.1. Let \mathcal{K} be a 2-category and $(T: \mathcal{K} \rightarrow \mathcal{K}, i: 1 \Rightarrow T, m: T^2 \Rightarrow T)$ a strict 2-monad on \mathcal{K} . A *pseudo-algebra* of T consists of

- i. a 0-cell A in \mathcal{K} ,
- ii. a 1-cell $\mathfrak{a}: TA \rightarrow A$ which we call structure map,
- iii. and invertible 2-cells $\zeta: 1_A \Rightarrow \mathfrak{a} \circ i_A$ and $\theta: \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$,

$$\begin{array}{ccc} \begin{array}{ccc} A & & \\ i_A \downarrow & \searrow 1 & \\ TA & \xrightarrow{\mathfrak{a}} & A \\ & \swarrow \zeta & \end{array} & \begin{array}{ccc} T^2 A & \xrightarrow{T\mathfrak{a}} & TA \\ m_A \downarrow & \theta \Downarrow & \downarrow \mathfrak{a} \\ TA & \xrightarrow{\mathfrak{a}} & A \end{array} & (1) \end{array}$$

subject to the following coherence axioms:

$$(\theta \bullet m_{TA}) \circ (\theta \bullet T^2 \mathfrak{a}) = (\theta \bullet Tm_A) \circ (\mathfrak{a} \bullet T\theta)$$

expressed by equality of pasting diagrams:

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$$\begin{array}{ccc}
 \begin{array}{c}
 T^3 A \xrightarrow{T^2 \mathbf{a}} T^2 A \xrightarrow{T \mathbf{a}} T A \\
 \downarrow m_{TA} \quad \downarrow m_A \quad \downarrow \theta \\
 T^2 A \xrightarrow{T \mathbf{a}} T A \xrightarrow{\mathbf{a}} A \\
 \downarrow m_A \quad \downarrow \theta \\
 T A \xrightarrow{\mathbf{a}} A
 \end{array}
 & = &
 \begin{array}{c}
 T^3 A \xrightarrow{T^2 \mathbf{a}} T^2 A \xrightarrow{T \mathbf{a}} T A \\
 \downarrow T m_A \quad \downarrow T \theta \\
 T^2 A \xrightarrow{T \mathbf{a}} T A \xrightarrow{\mathbf{a}} A \\
 \downarrow m_A \quad \downarrow \theta \\
 T A \xrightarrow{\mathbf{a}} A
 \end{array}
 \end{array} \quad (2)$$

and

$$(\theta \cdot T i_A) \circ (\mathbf{a} \cdot T \zeta) = id_{\mathbf{a}} = (\theta \cdot i_{TA}) \circ (\zeta \cdot \mathbf{a})$$

expressed by equality of pasting diagrams:

$$\begin{array}{ccc}
 \begin{array}{c}
 TA \xrightarrow{1_{TA}} TA \\
 \downarrow T i_A \quad \downarrow T \zeta \\
 T^2 A \xrightarrow{T \mathbf{a}} T A \\
 \downarrow m_A \quad \downarrow \theta \\
 T A \xrightarrow{\mathbf{a}} A
 \end{array}
 & = &
 \begin{array}{c}
 TA \xrightarrow{\mathbf{a}} A \\
 \downarrow 1_{TA} \quad \downarrow 1_A \\
 T A \xrightarrow{\mathbf{a}} A
 \end{array}
 & = &
 \begin{array}{c}
 TA \xrightarrow{\mathbf{a}} A \\
 \downarrow i_{TA} \quad \downarrow i_A \\
 T^2 A \xrightarrow{T \mathbf{a}} T A \xrightarrow{\zeta} T A \\
 \downarrow m_A \quad \downarrow \theta \\
 T A \xrightarrow{\mathbf{a}} A
 \end{array}
 \end{array} \quad (3)$$

DEFINITION 1.2. Suppose $(\mathbf{a}, \zeta_A, \theta_A) : TA \rightarrow A$ and $(\mathbf{b}, \zeta_B, \theta_B) : TB \rightarrow B$ are pseudo-algebras of a 2-monad T . A *lax morphism* from \mathbf{a} to \mathbf{b} consists of a 1-cell $f : A \rightarrow B$ and a 2-cell \check{f}

$$\begin{array}{ccc}
 TA & \xrightarrow{Tf} & TB \\
 \downarrow \mathbf{a} & \check{f} \Downarrow & \downarrow \mathbf{b} \\
 A & \xrightarrow{f} & B
 \end{array}$$

in such a way that

- $f \cdot \zeta_A = (\check{f} \cdot i_A) \circ (\zeta_B \cdot f)$ expressing the following pasting equality

$$\begin{array}{ccc}
 & & A \\
 & \swarrow i_A & \\
 TA & & \\
 \downarrow a & \zeta \uparrow & \downarrow 1 \\
 A & \xrightarrow{f} & B
 \end{array}
 =
 \begin{array}{ccccc}
 & & A & \xrightarrow{f} & B \\
 & \swarrow i_A & & \swarrow i_B & \\
 TA & \xrightarrow{Tf} & TB & & \\
 \downarrow a & \check{f} \Downarrow & \downarrow b & \zeta \uparrow & \downarrow 1 \\
 A & \xrightarrow{f} & B & &
 \end{array}$$

and

- $(f \cdot \theta_A) \circ (\check{f} \cdot T\mathfrak{a}) \circ (\mathfrak{b} \cdot Tf) = (\check{f} \cdot m_A) \circ (\theta_B \cdot T^2f)$ expressing the following pasting equality

$$\begin{array}{ccc}
 & T^2A & \xrightarrow{T^2f} & T^2B \\
 & \swarrow T\mathfrak{a} & \swarrow Tf & \swarrow T\mathfrak{b} \\
 TA & \xrightarrow{Tf} & TB & \\
 \downarrow a & \theta \Downarrow & \downarrow & \downarrow \\
 A & \xrightarrow{f} & B & \\
 & \swarrow \mathfrak{a} & \swarrow \check{f} & \swarrow \mathfrak{b}
 \end{array}
 =
 \begin{array}{ccc}
 & T^2A & \xrightarrow{T^2f} & T^2B \\
 & \downarrow m_A & & \downarrow m_B \\
 & TA & \xrightarrow{Tf} & TB \\
 & \swarrow \mathfrak{a} & \swarrow \check{f} & \swarrow \mathfrak{b} \\
 A & \xrightarrow{f} & B & \\
 & \downarrow \theta & & \downarrow \theta
 \end{array}$$

DEFINITION 1.3. A 2-monad $T : \mathcal{K} \rightarrow \mathcal{K}$ is said to be **lax idempotent** if given any two (pseudo) T -algebras $\mathfrak{a}: TA \rightarrow A$, $\mathfrak{b}: TB \rightarrow B$ and a 1-cell $f: A \rightarrow B$, there exists a unique 2-cell $\check{f}: \mathfrak{b} \circ Tf \Rightarrow f \circ \mathfrak{a}$ rendering (f, \check{f}) a lax morphism of pseudo T -algebras.

$$\begin{array}{ccc}
 TA & \xrightarrow{Tf} & TB \\
 \downarrow \mathfrak{a} & \check{f} \Downarrow & \downarrow \mathfrak{b} \\
 A & \xrightarrow{f} & B
 \end{array}$$

REMARK 1.4. Dually, reverse the direction of \check{f} in definition 1.3, then we get the notion of **co-lax idempotent** monad.

2 KZ-monads

DEFINITION 2.1. A 2-monad $T : \mathcal{K} \rightarrow \mathcal{K}$ is said to be **KZ-monad**¹ if $m \dashv i \cdot T$ in the 2-category $[\mathcal{K}, \mathcal{K}]$ with identity counit.

REMARK 2.2. Dual to the definition above, we define a monad T to be a **co-KZ-monad** by requiring $i \cdot T \dashv m$ with identity unit.

Suppose T is a co-KZ-monad and $i \cdot T \dashv m$. In particular unit of this adjunction is identity since $m \circ (i \cdot T) = 1$. Moreover, the identity 2-cell

$$\begin{array}{ccc} T & \xrightarrow{1} & T \\ \uparrow 1 & \text{id} \Downarrow & \uparrow m \\ T & \xrightarrow{T \cdot i} & T^2 \end{array}$$

has a mate

$$\begin{array}{ccc} T & \xrightarrow{1} & T \\ \downarrow 1 & \lambda \Downarrow & \downarrow i \cdot T \\ T & \xrightarrow{T \cdot i} & T^2 \end{array} \quad (4)$$

with properties $m \cdot \lambda = id_{1_T}$ and $\lambda \cdot i = id_{(T \cdot i) \circ i}$. These identity follow from triangle identities of adjunction $i \cdot T \dashv m$, and $(i \cdot T) \circ i = (T \cdot i) \circ i$ by naturality of i .

Suppose $\mathbf{a} : TA \rightarrow A$ is a pseudo algebra for T . We would like to calculate the composite 2-cell

$$\begin{array}{ccccc} & \overset{i_{TA}}{\curvearrowright} & & \overset{\mathbf{a} \circ T\mathbf{a}}{\curvearrowright} & \\ TA & \xrightarrow{\quad} & T^2A & \xrightarrow{\quad} & TA \\ & \Downarrow \lambda_A & & \Downarrow \theta & \\ & \underset{Ti_A}{\curvearrowleft} & & \underset{\mathbf{a} \circ m_A}{\curvearrowleft} & \end{array}$$

In the diagram below, since $m_A \circ \lambda_A = id$, the left column of 2-cells collapses to identity, and therefore we have

$$\begin{array}{ccc} TA & \xrightarrow{1} & TA & \xrightarrow{\mathbf{a}} & A \\ \downarrow 1 & \lambda \Downarrow & \downarrow i_{TA} & \downarrow i_A & \\ TA & \xrightarrow{T i_A} & T^2A & \xrightarrow{T \mathbf{a}} & TA & \xleftarrow{\zeta} & 1_A \\ \downarrow 1 & & \downarrow m_A \theta \Downarrow & \downarrow \mathbf{a} & & & \\ TA & \xrightarrow{1} & TA & \xrightarrow{\mathbf{a}} & A & \xleftarrow{\zeta} & 1_A \end{array} = \begin{array}{ccc} TA & \xrightarrow{\mathbf{a}} & A \\ \downarrow 1_{TA} & & \downarrow 1_A \\ TA & \xrightarrow{\mathbf{a}} & A \end{array}$$

$$\theta \cdot \lambda_A = \zeta^{-1} \cdot \mathbf{a}$$

¹KZ: short for ‘Kock-Zöberlein’

On the other hand, we can compose row-wise instead, and we get

$$\theta \cdot \lambda_A = (\theta \cdot T i_A) \circ (\mathbf{a} \circ T \mathbf{a} \cdot \lambda_A) = (\mathbf{a} \cdot T \zeta^{-1}) \circ (\mathbf{a} \circ T \mathbf{a} \cdot \lambda_A)$$

Thus, in the end, we have

$$\begin{array}{c} \begin{array}{ccc} TA & \xrightarrow{i_{TA}} & T^2 A \xrightarrow{T\mathbf{a}} TA \xrightarrow{\mathbf{a}} A \\ \lambda_A \Downarrow & & \\ TA & \xrightarrow{T i_A} & T^2 A \xrightarrow{T\zeta^{-1}} TA \xrightarrow{\mathbf{a}} A \\ \text{1} & & \end{array} \\ = \\ \begin{array}{ccc} TA & \xrightarrow{\mathbf{a}} & A \\ i_A \nearrow & & \searrow \mathbf{a} \\ & & TA \xrightarrow{\zeta^{-1}} A \\ & & \text{1} \end{array} \end{array} \quad (5)$$

LEMMA 2.3. Let T be a KZ-monad, and A an object of \mathcal{K} . Then any pseudo T -algebra on A is left adjoint to unit i_A . Conversely, if i_A has a left adjoint with invertible counit then this left adjoint is a pseudo T -algebra.

REMARK 2.4. This observation requires a bit of conceptual explanation: for a KZ-monad T , any object admits at most one pseudo T -algebra structure, up to unique isomorphism. So a KZ-monad is a nicely-behaved 2-monad whose algebras are ‘property-like’ in the sense that the structure is a (reflective) left adjoint to the unit. Similarly, for a co-KZ-monad T the structure \mathbf{a} is right adjoint to the unit i_A and the invertible unit of this adjunction is given by $\zeta: 1 \Rightarrow \mathbf{a} i_A$ in diagram (1).

$$\begin{array}{ccc} & i_A & \\ & \curvearrowright & \\ TA & \xleftarrow{\quad} & A \\ & \perp & \\ & \curvearrowleft & \\ & \mathbf{a} & \end{array}$$

What about counit of $i_A \dashv \mathbf{a}$? Here is a calculation² of counit using mate λ_A introduced in diagram 4.

$$\begin{array}{c} \begin{array}{ccc} & \mathbf{a} & \\ & \curvearrowright & \\ TA & \xrightarrow{\quad} & A \xrightarrow{i_A} TA \\ i_{TA} \nearrow & & \searrow \mathbf{a} \\ & & T^2 A \xrightarrow{T\mathbf{a}} TA \\ \lambda_A \Downarrow & & \\ TA & \xrightarrow{T i_A} & T^2 A \xrightarrow{T\zeta^{-1}} TA \\ \text{1} & & \end{array} \end{array} \quad (6)$$

We prove the triangle identities of adjunction with the proposed unit and counit:

$$\begin{aligned} (\mathbf{a} \cdot T \zeta^{-1} \circ (T \mathbf{a} \cdot \lambda_A)) \circ (\zeta \cdot \mathbf{a}) &= (\zeta^{-1} \cdot \mathbf{a}) \circ (\zeta \cdot \mathbf{a}) \quad \{\text{by equality of pasting diagrams (5)}\} \\ &= id_{\mathbf{a}} \quad \{\text{factoring out } \mathbf{a}\} \end{aligned}$$

²The dual of this situation, i.e. unit in the case of KZ-monad, is calculated in page 112 of (Street 1974).

Also,

$$\begin{aligned}
 ((T\zeta^{-1} \circ (T\mathbf{a} \cdot \lambda_A)) \cdot i_A) \circ (i_A \cdot \zeta) &= (T\zeta^{-1} \cdot i_A) \circ (i_A \cdot \zeta) && \{\lambda_A \cdot i_A = id\} \\
 &= (i_A \cdot \zeta^{-1}) \circ (i_A \cdot \zeta) && \{2\text{-naturality of } i: 1 \Rightarrow T\} \\
 &= id_{i_A} && \{\text{factoring out } i_A\}
 \end{aligned}$$

In (Street 1974), we also see a converse of remark above.

LEMMA 2.5. Suppose $T: \mathcal{K} \rightarrow \mathcal{K}$ is a co-KZ 2-monad and suppose a 0-cell A , a 1-cell $\mathbf{a}: TA \rightarrow A$, and an isomorphism 2-cell $\zeta: 1 \Rightarrow \mathbf{a} \circ i_A$ are given in \mathcal{K} , and furthermore, ζ^{-1} satisfies pasting equality (5). We have:

1. ζ is the unit for an adjunction $i_A \dashv \mathbf{a}$ whose counit ε is given by $(T\zeta^{-1}) \circ (T\mathbf{a} \cdot \lambda_A)$ (composite 2-cell in diagram (6)).
2. The 2-cell $\theta: \mathbf{a} \circ T\mathbf{a} \Rightarrow \mathbf{a} \circ m_A$, obtained by taking double mate of $\lambda_A \cdot i_A = id$, is an iso 2-cell.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 T^2A & \xleftarrow{T i_A} & TA \\
 \uparrow i_{TA} & id \uparrow & \uparrow i_A \\
 TA & \xleftarrow{i_A} & A
 \end{array} & \rightsquigarrow & \begin{array}{ccc}
 T^2A & \xrightarrow{T \mathbf{a}} & TA \\
 \downarrow m_A & \theta \downarrow & \downarrow \mathbf{a} \\
 TA & \xrightarrow{\mathbf{a}} & A
 \end{array}
 \end{array}$$

3. 2-cell θ enriches (A, \mathbf{a}, ζ) with the structure of a pseudo T -algebra.

PROPOSITION 2.6. Any KZ-monad (resp. co-KZ-monad) is lax idempotent (resp. co-lax idempotent). (Street 1974) (Kock 1995)

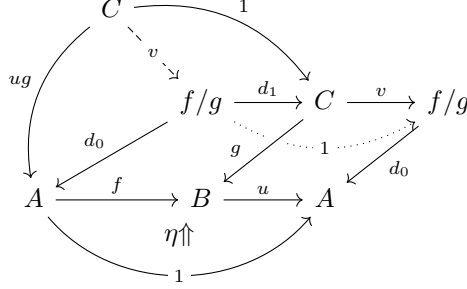
3 A general useful lemma in 2-categories

There is an innocent looking yet quite important proposition in (Street 1974) which may be overlooked in first reading of the paper.³ This is proposition 6 in that paper. We state it here.

PROPOSITION 3.1. Suppose $f: A \rightarrow B$ is a 1-cell with right adjoint u , unit η , and counit ϵ in a 2-category \mathcal{K} with comma objects. For any 1-cell $g: C \rightarrow B$, the unique filling arrow $v: C \rightarrow f/g$ obtained by factoring $\epsilon \cdot g$ through (strict) comma square $\langle f/g, d_0, d_1, \phi \rangle$ is right adjoint to d_1 with counit identity.

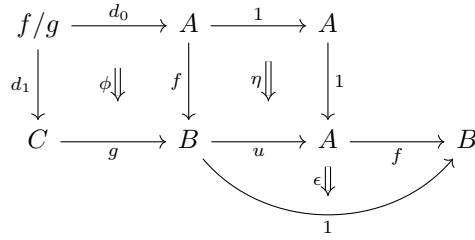
³Unfortunately, this occurred in the case of author.

The 1-cell v in the proposition is uniquely determined by equations $d_1v = 1$, $d_0v = ug$, and $\phi \cdot v = \epsilon \cdot g$. Moreover, the proposition states that we can lift the 2-cell η in the lower part of the diagram to a 2-cell $1 \Rightarrow vd_1$ in the upper part.



Proof. We first construct the would-be unit β of adjunction $d_1 \dashv v$. Using the fact $(\epsilon \cdot f) \circ (f \cdot \eta) = 1$ in chasing the diagram below, we obtain:

$$(\phi \cdot vd_1) \circ (fu \cdot \phi) \circ (f \cdot \eta \cdot d_0) = (\epsilon \cdot gd_1) \circ (fu \cdot \phi) \circ (f \cdot \eta \cdot d_0) = \phi$$



We (uniquely) define $\tau_1: 1 \Rightarrow vd_1$ to be the unique 2-cell with

$$\begin{aligned} d_0 \cdot \tau_1 &= (u \cdot \phi) \circ (\eta \cdot d_0) \\ d_1 \cdot \tau_1 &= 1 \end{aligned} \quad (7)$$

One readily verifies that with id and τ_1 , d_1 and v satisfy triangle equations of adjunction. \square

REMARK 3.2. A useful special case of above proposition is when f and g are both identity 1-cells $1: E \rightarrow E$. In that case $f/g \simeq E^{\rightarrow}$ and $v = i_E$. The unit $\tau_1: 1_{E^{\rightarrow}} \Rightarrow i_E \circ e_1$ is the unit of familiar adjunction $e_1 \dashv i_E$. In the case when 2-category \mathcal{K} is 2-category of (small) categories, $\tau_1(u) = (u, 1)$ for any $u: b_0 \rightarrow b_1$ in E^{\rightarrow} .

$$\begin{array}{ccc} e_0 & \xrightarrow{u} & e_1 \\ u \downarrow & & \downarrow 1 \\ e_1 & \xrightarrow{1} & e_1 \end{array}$$

similarly, the dual of proposition 3.1 when applied to $f = g = 1$ gives i_E as left adjoint of $e_0: E^{\rightarrow} \rightarrow E$. The unit of this adjunction is identity, making e_0 a retraction. The counit is given by the unique 2-cell $\tau_0: i_E \circ e_0 \Rightarrow 1_{E^{\rightarrow}}$ defined by the equations $e_0\tau_0 = 1$ and $e_1\tau_0 = \phi$. In particular, in 2-category of small categories we have $\tau_0(u) = (1, u)$.

4 Fibrations as pseudo-algebras of a co-KZ-monad

Let \mathcal{K} be a representable 2-category. Define \mathcal{K}/B to be the strict slice 2-category over B , meaning the morphism triangles commute up to equality. (Street 1974) constructs KZ-monads $L, R: \mathcal{K}/B \rightrightarrows \mathcal{K}/B$. The idea is, for a morphism $p: E \rightarrow B$, an algebra $R(p) \rightarrow p$ (resp. $L(p) \rightarrow p$) if it exist, corresponds to the fibration structure on p (resp. opfibration structure). We will only present explicit construction and calculation for the case of fibration⁴ and thus, we will mainly concern ourselves with 2-monad R . However, when necessary, we will comment on the dual results for the case of opfibrations. We now define 2-monad R : It takes an object (E, p) to $(B/p, R(p))$ where

$$\begin{array}{ccc} B/p & \xrightarrow{\hat{d}_1} & E \\ R(p) \downarrow & \phi_p \uparrow & \downarrow p \\ B & \xrightarrow{1} & B \end{array} \quad (8)$$

is a comma square.

REMARK 4.1. 2-cell ϕ_p can be constructed as follows:

$$\begin{array}{ccc} B/p & \xrightarrow{\hat{d}_1} & E \\ R(p) \downarrow & \phi_p \uparrow & \downarrow p \\ B & \xrightarrow{1} & B \end{array} = \begin{array}{ccc} B/p & \xrightarrow{\hat{d}_1} & E \\ \hat{p} \downarrow \lrcorner & & \downarrow p \\ B & \xrightarrow{d_1} & B \\ d_0 \downarrow & \phi \uparrow & \downarrow 1 \\ B & \xrightarrow{1} & B \end{array}$$

The action of R on morphisms is given as follows:

If $f: (E', p') \rightarrow (E, p)$ is a 1-cell in \mathcal{K}/B , then define $R(f)$ to be the unique 1-cell with $\hat{d}_1 \circ R(f) = f \circ \hat{d}'_1$ and $\hat{p} \circ R(f) = \hat{p}'$.

$$\begin{array}{ccc} B/p' & \xrightarrow{\hat{d}'_1} & E' \\ R(f) \downarrow \lrcorner & & \downarrow f \\ B/p & \xrightarrow{\hat{d}_1} & E \\ \hat{p} \downarrow \lrcorner & & \downarrow p \\ B & \xrightarrow{d_1} & B \end{array}$$

(Curved arrows labeled \hat{p}' and p' connect B/p' to B and E' to B respectively.)

⁴Unlike Street's paper whereby he works with opfibration structures and as a result, he chooses to work with 2-monad L on \mathcal{K}/B which takes p to $L(p) := p/B$.

Similarly if $\sigma: f \Rightarrow g$ is a 2-cell in \mathcal{K}/B , then we have a unique induced 2-cell $R(\sigma): R(f) \Rightarrow R(g)$ with $\hat{d}_1 \cdot R(\sigma) = \sigma \cdot \hat{d}'_1$ and $\hat{p} \cdot R(\sigma) = id_{\hat{p}}$.

PROPOSITION 4.2. 2-functor $R: \mathcal{K}/B \rightarrow \mathcal{K}/B$ is a 2-monad.

The unit of monad $i: id \Rightarrow R$ at (E, p) is given by the unique arrow $i(p): E \rightarrow B/p$ with property that $R(p) \circ i(p) = p$ and $\hat{d}_1 \circ i(p) = 1_E$, and moreover $\phi_p \cdot i(p) = id_p$, all inferred by universal property of comma object B/p .

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & \curvearrowright & & \\
 E & \xrightarrow{i(p)} & B/p & \xrightarrow{\hat{d}_1} & E \\
 & \searrow p & \downarrow R(p) & \uparrow \phi_p & \downarrow p \\
 & & B & \xrightarrow{1} & B
 \end{array}$$

It also follows that $\hat{d}_1 \dashv i(p)$ with identity counit. Indeed, $i(p)$ is v in proposition 3.1, when $f = 1$ and $g = p$. From there, we also get the unit $\tau_1(p)$ of adjunction with $R(p) \cdot \tau_1(p) = \phi_p$.

The multiplication $m: R^2 \Rightarrow R$ of monad at 0-cell (E, p) is given by the unique arrow $m(p): B/R(p) \rightarrow B/p$

$$\begin{array}{ccccc}
 & & m(p) & & \\
 & & \curvearrowright & & \\
 B/R(p) & \xrightarrow{\widehat{d}_1} & B/p & \xrightarrow{\hat{d}_1} & E \\
 \hat{p} \downarrow & \lrcorner & \downarrow \hat{p} & \lrcorner & \downarrow p \\
 B \rightarrow & \xrightarrow{\widehat{d}_1} & B \rightarrow & \xrightarrow{d_1} & B \\
 d_0 \downarrow & \lrcorner & \downarrow d_0 & \lrcorner & \downarrow 1 \\
 B \rightarrow & \xrightarrow{d_1} & B & \xrightarrow{1} & B \\
 d_0 \downarrow & \lrcorner & \downarrow 1 & \lrcorner & \\
 B & \xrightarrow{1} & B & &
 \end{array} \tag{9}$$

with the property that $R(p) \circ m(p) = R^2(p)$ and $\hat{d}_1 \circ m(p) = \hat{d}_1 \circ \widehat{d}_1$, and moreover $\phi_p \cdot m(p) = (\phi_p \cdot \widehat{d}_1) \circ (\phi_p \cdot d_0 \cdot \hat{p}) = (\phi_p \cdot \widehat{d}_1) \circ \phi_{R(p)}$, all inferred by universal property of comma object B/p .

PROPOSITION 4.3. 2-monad $R: \mathcal{K}/B \rightarrow \mathcal{K}/B$ is a co-KZ-monad.

Proof. We have to show that $i \cdot R \dashv m$. □

Now, we would like to see what a pseudo algebra $\mathfrak{a}: R(p) \rightarrow p$ in \mathcal{K}/B looks like. The

fact that \mathbf{a} is a morphism in \mathcal{K}/B provides us with a morphism \mathbf{a} which makes the diagram

$$\begin{array}{ccc} B/p & \xrightarrow{\mathbf{a}} & E \\ & \searrow R(p) & \swarrow p \\ & & B \end{array} \quad (10)$$

commute. Moreover, being a co-KZ-monad, R generates an adjunction $i(p) \dashv \mathbf{a}$ whose unit is the invertible 2-cell $\zeta: 1 \Rightarrow \mathbf{a} \circ i(p)$ by remark 2.4. The counit ε is given as $R\zeta^{-1} \circ (R\mathbf{a} \cdot \lambda_p)$. Whiskering with \hat{d}_1 yields a 2-cell $\hat{d}_1 \cdot \varepsilon: \mathbf{a} \Rightarrow \hat{d}_1$

Observe that $\hat{d}_1 \cdot \varepsilon = \hat{d}_1$

$$\begin{array}{ccccc} & & 1 & & \\ & & \Downarrow \zeta & & \\ E & \xrightarrow{i(p)} & B/p & \xrightarrow{\mathbf{a}} & E \\ & \searrow p & \downarrow R(p) & \swarrow p & \\ & & \mathcal{A} & & \end{array} \quad (11)$$

such that $p \cdot \zeta = id_p$. The counit of this

In the example below we investigate how the construction above look like when we choose 2-category of (locally small) categories as our working 2-category.

EXAMPLE 4.4. Let's take $\mathcal{K} = \mathbf{Cat}$ to be the strict 2-category of categories, functors, and natural transformations. First and foremost, for a functor $p: E \rightarrow B$, the comma category B/p is given as a category whose objects are pairs $\langle f: b \rightarrow p(e), e \rangle$ where f is morphism in B :⁵

$$\begin{array}{ccc} & e & \\ & \downarrow p & \\ b_0 & \xrightarrow{f} & b_1 \end{array}$$

Morphisms of B/p are of the form

$$\begin{array}{ccccc} & e & & \tilde{h}_1 & \\ & \downarrow p & & \searrow & e' \\ b_0 & \xrightarrow{f} & b_1 & & \downarrow p \\ & \searrow h_0 & c_0 & \xrightarrow{g} & c_1 \\ & & & \swarrow h_1 & \end{array}$$

Functor $R(p)$ as in diagram (8) takes pair $\langle f, e \rangle$ to $b_0 = \text{dom}(f)$, and \hat{d}_1 is simply the second projection; it takes $\langle f, e \rangle$ to e . The unit of monad R at (E, p) , i.e. $i(p): E \rightarrow B/p$, takes an object e of E to the object

⁵ $e \mapsto b_1$ indicates that $p(e) = b_1$.

$$\begin{array}{c}
 e \\
 \downarrow p \\
 p(e) \equiv p(e)
 \end{array}$$

and $\tau_1(p): 1_{B/p} \Rightarrow i(p) \circ \hat{d}_1$ induces a morphisms $B/p \rightarrow B/p^{\rightarrow}$ which takes an object of B/p in above to

$$\begin{array}{ccc}
 & e_1 & \\
 & \downarrow p & \parallel \\
 b_0 & \xrightarrow{f} & b_1 & \xrightarrow{e_1} & b_1 \\
 & \searrow f & & \parallel & \downarrow p \\
 & & b_1 & \equiv & b_1
 \end{array}$$

We also note that $\widehat{d}_1^{\rightarrow}$ (as in diagram 9) is given by the action

$$\begin{array}{ccc}
 & e & \\
 & \downarrow & \\
 b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 & \mapsto & b_1 \xrightarrow{g} b_2 \\
 & & \downarrow e
 \end{array}$$

and multiplication $m(p)$ given by

$$\begin{array}{ccc}
 & e & \\
 & \downarrow & \\
 b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 & \mapsto & b_0 \xrightarrow{g \circ f} b_2 \\
 & & \downarrow e
 \end{array}$$

Now, suppose that $\mathbf{a}: R(p) \rightarrow p$ is a pseudo algebra for 2-monad R . By commutativity of diagram 10 we know that $p(\mathbf{a}\langle f, e \rangle) = \text{dom}(f)$. So we draw

$$\begin{array}{c}
 \mathbf{a}\langle f, e \rangle \\
 \downarrow p \\
 b_0 \xrightarrow{f} b_1
 \end{array}$$

As observed in diagram 11 we get an isomorphism lift of identity in the base:

$$\begin{array}{ccc}
 e & \xrightarrow{\zeta(e)} & \mathbf{a}\langle 1_{p(e)}, e \rangle \\
 \downarrow p & & \downarrow p \\
 p(e) & \equiv & p(e)
 \end{array}$$

Observe that functors $R(i(p)): B/p \rightarrow B/R(p)$ and $i(R(p)): B/p \rightarrow B/R(p)$ are given as follows:

$$R(i(p)) : \begin{array}{ccc} & e & \\ & \downarrow & \\ b_0 \xrightarrow{f} & b_1 & \\ & & \end{array} \quad \mapsto \quad \begin{array}{ccc} & e & \\ & \downarrow & \\ b_0 \xrightarrow{f} & b_1 & \equiv b_1 \\ & & \end{array}$$

and

$$i(R(p)) : \begin{array}{ccc} & e & \\ & \downarrow & \\ b_0 \xrightarrow{f} & b_1 & \\ & & \end{array} \quad \mapsto \quad \begin{array}{ccc} & e & \\ & \downarrow & \\ b_0 \equiv & b_0 \xrightarrow{f} & b_1 \\ & & \end{array}$$

and the mate 2-cell λ as in diagram (4) appears as a natural transformations in this case where $\lambda_p : i(R(p)) \Rightarrow R(i(p))$ can be illustrated as

$$\begin{array}{ccccc} & & & e & \\ & & & \downarrow & \\ & & & p & \\ & & & \Downarrow & \\ & & & e & \\ & & & \downarrow & \\ & & & p & \\ & & & \Downarrow & \\ & & & b_1 & \\ & & & \downarrow & \\ & & & p & \\ & & & \Downarrow & \\ & & & b_1 & \\ & & & \downarrow & \\ & & & p & \\ & & & \Downarrow & \\ & & & b_1 & \\ & & & \downarrow & \\ & & & p & \\ & & & \Downarrow & \\ & & & b_1 & \\ & & & \downarrow & \\ & & & p & \\ & & & \Downarrow & \\ & & & b_1 & \end{array}$$

We keep in mind that $R(\mathbf{a}) \circ R \cdot i(p)\langle f, e \rangle = \langle f, \mathbf{a}\langle 1_{b_1}, e \rangle \rangle$, and hence $R(\zeta)\langle f, e \rangle$ is illustrated in below:

$$\begin{array}{ccc} & e & \\ & \downarrow & \\ & p & \\ & \Downarrow & \\ & e & \\ & \downarrow & \\ & p & \\ & \Downarrow & \\ & b_1 & \\ & \downarrow & \\ & p & \\ & \Downarrow & \\ & b_1 & \\ & \downarrow & \\ & p & \\ & \Downarrow & \\ & b_1 & \end{array} \quad \begin{array}{l} \zeta(e) \\ \searrow \\ \mathbf{a}\langle 1_{b_1}, e \rangle \end{array} \quad (12)$$

In addition, invertible 2-cell $\theta(p) : \mathbf{a} \circ R(\mathbf{a}) \Rightarrow \mathbf{a} \circ m(p)$ provides us with an isomorphism $\mathbf{a}\langle f, \mathbf{a}\langle g, e \rangle \rangle \rightarrow \mathbf{a}\langle gf, e \rangle$. Now, we study coherence equations (2) and (3) in our case, which state that for any morphism $f : b_0 \rightarrow b_1$ together with any object e in E over b_1 , the following diagram (in the fibre over b_0) commute :

$$\begin{array}{ccc} \mathbf{a}\langle f, e \rangle & \xrightarrow{\mathbf{a}\cdot R(\zeta)} & \mathbf{a}\langle f, \mathbf{a}\langle 1_{b_1}, e \rangle \rangle \\ \downarrow \zeta \cdot \mathbf{a} & \Downarrow & \downarrow \theta \cdot R(i(p)) \\ \mathbf{a}\langle 1_{b_0}, \mathbf{a}\langle f, e \rangle \rangle & \xrightarrow{\theta \cdot i(R(p))} & \mathbf{a}\langle f \circ 1_{b_0}, e \rangle \end{array}$$

and, for every chain of morphisms $b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 \xrightarrow{h} b_3$ in B and any object e in E over b_3 , the diagram (in the fibre over b_0)

$$\begin{array}{ccc} \mathbf{a}\langle f, \mathbf{a}\langle g, \mathbf{a}\langle h, e \rangle \rangle \rangle & \longrightarrow & \mathbf{a}\langle gf, \mathbf{a}\langle h, e \rangle \rangle \\ \downarrow & & \downarrow \\ \mathbf{a}\langle f, \mathbf{a}\langle gh, e \rangle \rangle & \longrightarrow & \mathbf{a}\langle hgf, e \rangle \end{array}$$

commutes. Finally, the counit of adjunction $i(p) \dashv \mathbf{a}$, as computed in diagram 6, gives us the lift $\tilde{f} = \hat{d}_1 \cdot \varepsilon = \hat{d}_1 \cdot (R\zeta^{-1} \circ (R\mathbf{a} \cdot \lambda_p))$ of f :

$$\begin{array}{ccccc} & \mathbf{a}\langle f, e \rangle & & & \\ & \downarrow p & \searrow \hat{d}_1(R\mathbf{a} \cdot \lambda_p) & & \\ & b_0 & & \mathbf{a}\langle 1_{b_1}, e \rangle & \\ & \downarrow p & & \downarrow p & \searrow \hat{d}_1(R\zeta^{-1}) \\ b_0 & \xrightarrow{f} & b_1 & & e \\ \downarrow 1 & & \downarrow p & & \downarrow p \\ b_0 & \xrightarrow{f} & b_1 & & \\ \downarrow & & \downarrow & & \\ b_0 & \xrightarrow{f} & b_1 & & \end{array}$$

It remains to prove that \tilde{f} as defined is cartesian. One can try to prove this directly. However, we prove this in a more general setting in the next section.

5 Chevalley criterion

Suppose p is a 0-cell in \mathcal{K}/B . There is a unique derived 1-cell Γ_1 with properties $R(p)\Gamma_1 = d_0 p \vec{}$, $\hat{d}_1 \Gamma_1 = e_1$, and $\phi_p \cdot \Gamma_1 = p \cdot \phi_E$.

$$\begin{array}{ccccc} E \vec{} & & & & \\ \downarrow p \vec{} & \searrow \Gamma_1 & & & \\ B \vec{} & & B/p & \xrightarrow{\hat{d}_1} & E \\ & \searrow d_0 & \downarrow R(p) & \uparrow \phi_p & \downarrow p \\ & & B & \xrightarrow{1} & B \end{array}$$

The lemma below will be crucial in certain calculations of 2-cells in the proof of proposition

LEMMA 5.1. We have $\hat{d}_1 \Gamma_1 \cdot \tau_0 = \phi_E$ and $R(p)\Gamma_1 \cdot \tau_0 = id_{R(p)\Gamma_1}$ and from these it follows that $(\tau_1(p) \cdot \Gamma_1) \circ (\Gamma_1 \cdot \tau_0) = i(p) \cdot \phi_E$, by 2-dimensional universal property of B/p . Also, $\hat{d}_1 \Gamma_1 \cdot \tau_1 = id_{e_1}$ and $R(p)\Gamma_1 \cdot \tau_1 = p \cdot \phi_E$ and it follows that $\tau_1(p) \cdot \Gamma_1 = \Gamma_1 \cdot \tau_1$.

Proof. The first identity holds since $e_1 \cdot \tau_0 = \phi_E$ due to universal property of comma object E^{\rightarrow} . For the second identity observe that $R(p)\Gamma_1 \cdot \tau_0 = pe_0 \cdot \tau_0 = id_{pe_0} = id_{R(p)\Gamma_1}$, by one of triangle identity of adjunction $i_E \dashv e_0$. Now, notice that

$$\begin{aligned} \hat{d}_1[(\tau_1(p) \cdot \Gamma_1) \circ (\Gamma_1 \cdot \tau_0)] &= \phi_E = \hat{d}_1[i(p) \cdot \phi_E] \\ R(p)[(\tau_1(p) \cdot \Gamma_1) \circ (\Gamma_1 \cdot \tau_0)] &= R(p) \cdot \tau_1(p) \cdot \Gamma_1 = \phi_p \cdot \Gamma_1 = p \cdot \phi_E = R(p)[i(p) \cdot \phi_E] \end{aligned}$$

To prove the second claim, similar to the first case, we appeal to the universal property of B/p with respect to incoming 2-cell, together with following identities of 2-cells:

$$\begin{aligned} \hat{d}_1 \cdot \tau_1(p) \cdot \Gamma_1 &= id_{\hat{d}_1\Gamma_1} = id_{e_1} = \Gamma_1 \cdot \tau_1 \\ R(p) \cdot \tau_1(p) \cdot \Gamma_1 &= \phi_p \cdot \Gamma_1 = \phi_p \cdot \Gamma_1 = p \cdot \phi_E = R(p)\Gamma_1 \cdot \tau_1 \end{aligned}$$

□

DEFINITION 5.2. We say a 1-cell $p: E \rightarrow B$ in \mathcal{K} satisfies **Chevalley criterion** if Γ_1 has a right adjoint Λ_1 in \mathcal{K}/B with isomorphism counit. Sometimes we call such an adjunction $\Gamma_1 \dashv \Lambda_1$ a Chevalley adjunction.

PROPOSITION 5.3. Given 1-cell $\Gamma_1: E^{\rightarrow} \rightarrow B/p$ as defined before lemma 5.1, we have a bijection between collection of 1-cells $p: E \rightarrow B$ equipped with an R -pseudo algebra $(\mathfrak{a}, \zeta, \theta)$ and collection of Chevalley adjoints $(\Gamma_1, \Lambda_1, \epsilon, \eta)$. Moreover, pseudo-algebra is normalized if and only if counit ϵ is identity.

Proof. Given a pseudo algebra $\mathfrak{a}: R(p) \rightarrow p$, we construct a right adjoint Λ_1 and show that the counit of adjunction is isomorphism. Hence p satisfies Chevalley criterion. Note that the unit $\tau_1(p)$ of adjunction $\hat{d}_1 \dashv i(p)$ defines a unique 1-cell $k: B/p \rightarrow (B/p)^{\rightarrow}$ obtained by factoring $\tau_1(p)$ through comma square $\langle (B/p)^{\rightarrow}, \pi_0, \pi_1, \phi_{B/p} \rangle$. Thus, $\pi_0 k = 1_{B/p}$ and $\pi_1 k = i(p)\hat{d}_1$, and $\phi_{B/p} \cdot k = \tau_1(p)$. Define $\Lambda_1 := \mathfrak{a}^{\rightarrow} \circ k$. We note that

$$\begin{aligned} e_0 \Lambda_1 &= e_0 \mathfrak{a}^{\rightarrow} k && \{\text{definition of } \Lambda_1\} \\ &= \mathfrak{a} \pi_0 k && \{\text{definition of } \mathfrak{a}^{\rightarrow}\} \\ &= \mathfrak{a} && \{\text{definition of } k\} \end{aligned} \tag{13}$$

This establishes that Λ_1 is indeed a 1-cell in \mathcal{K}/B , since $d_0 p^{\rightarrow} \Lambda_1 = pe_0 \Lambda_1 = p\mathfrak{a} = R(p)$. Also, a diagram chase shows that the front square in the diagram below commutes:

$$\begin{aligned} \hat{d}_1 \Gamma_1 \Lambda_1 &= e_1 \Lambda_1 && \{\text{definition of } \Gamma_1\} \\ &= e_1 \mathfrak{a}^{\rightarrow} k && \{\text{definition of } \Lambda_1\} \\ &= \mathfrak{a} \pi_1 k && \{\text{definition of } \mathfrak{a}^{\rightarrow}\} \\ &= \mathfrak{a} i(p) \hat{d}_1 && \{\text{definition of } k\} \end{aligned} \tag{14}$$

$$\begin{array}{ccc}
 & (B/p)^{\rightarrow} & \\
 & \uparrow k & \swarrow \pi_i \\
 B/p & \xrightarrow{\hat{d}_1} & E \\
 \downarrow \Gamma_1 \Lambda_1 & \searrow \Lambda_1 & \downarrow i(p) \\
 & E^{\rightarrow} & B/p \\
 & \swarrow \Gamma_1 & \downarrow \mathbf{a} \\
 B/p & \xrightarrow{\hat{d}_1} & E \\
 & \searrow e_i & \\
 & & E
 \end{array} \quad (i = 1, 2)$$

We also note that

$$\begin{aligned}
 R(p)\Gamma_1\Lambda_1 &= d_0 p^{\rightarrow} \Lambda_1 = R(p) \\
 \phi_p \cdot (\Gamma_1\Lambda_1) &= p \cdot \phi_E \cdot \Lambda_1 = p\mathbf{a} \cdot \phi_{B/p} \cdot k = p\mathbf{a} \cdot \tau_1(p) = R(p) \cdot \tau_1(p) = \phi_p
 \end{aligned} \quad (15)$$

Equations (14) and (15), and definition of $R(\mathbf{a}i(p))$ altogether prove that

$$\Gamma_1 \circ \Lambda_1 = R(\mathbf{a} \circ i(p)) = R(\mathbf{a}) \circ R(i(p))$$

and we shall show that counit $\epsilon: \Gamma_1 \circ \Lambda_1 \Rightarrow 1$ is given by $R(\zeta^{-1})$ which is invertible.⁶ Also notice that $p\hat{d}_1 \cdot \epsilon = p\hat{d}_1 \cdot R(\zeta^{-1}) = p \cdot \zeta^{-1} \cdot \hat{d}_1 = id_{p\hat{d}_1}$, and $R(p) \cdot \epsilon = R(p) \cdot R(\zeta^{-1}) = id_{R(p)}$. This guarantees that the counit lives in \mathcal{K}/B . Moreover, definition of $R(\zeta)$ implies that $\phi_p \cdot \epsilon = \phi_p$. Now, we propose the unit; define the 2-cell $\eta: 1 \Rightarrow \Lambda_1 \circ \Gamma_1$ to be the unique 2-cell with

$$\begin{aligned}
 e_0 \cdot \eta &= (\mathbf{a}\Gamma_1 \cdot \tau_0) \circ (\zeta \cdot e_0) \\
 e_1 \cdot \eta &= \zeta \cdot e_1
 \end{aligned} \quad (16)$$

Note that the vertical composition of 2-cells in (16) makes sense since $\mathbf{a}i(p)e_0 = \mathbf{a}\Gamma_1 i_E e_0$ which holds as one can easily see that $\Gamma_1 i_E = i(p)$. Furthermore, $e_0 \cdot \eta$ and $e_1 \cdot \eta$ are compatible in the sense that

$$\begin{aligned}
 (\phi_E \cdot \Lambda_1 \Gamma_1) \circ (e_0 \eta) &= (\phi_E \cdot \mathbf{a}^{\rightarrow} k \Gamma_1) \circ (e_0 \eta) && \{\text{definition of } \Lambda_1\} \\
 &= (\mathbf{a}\phi_{B/p} \cdot k \Gamma_1) \circ (e_0 \eta) && \{\text{definition of } \mathbf{a}^{\rightarrow}\} \\
 &= (\mathbf{a}\tau_1(p) \cdot \Gamma_1) \circ (e_0 \eta) && \{\text{definition of } k\} \\
 &= (\mathbf{a}\tau_1(p) \cdot \Gamma_1) \circ (\mathbf{a}\Gamma_1 \cdot \tau_0) \circ (\zeta \cdot e_0) && \{\text{substituting } e_0 \cdot \eta\} \\
 &= \mathbf{a}(\tau_1(p) \cdot \Gamma_1 \circ \Gamma_1 \cdot \tau_0) \circ (\zeta \cdot e_0) && \{\text{factoring out } \mathbf{a}\} \\
 &= (\mathbf{a}i(p) \cdot \phi_E) \circ (\zeta \cdot e_0) && \{\text{Lemma 5.1}\} \\
 &= (\zeta \cdot e_1) \circ \phi_E && \{\text{exchange rule}\} \\
 &= (e_1 \eta) \circ (\phi_E) && \{\text{substituting } e_1 \cdot \eta\}
 \end{aligned}$$

⁶When $\mathcal{K} = \mathbf{Cat}$, $R(\zeta)$ is illustrated in diagram 12.

In the next step, we prove that proposed unit⁷ η and counit ϵ satisfy triangle equations of adjunction. To prove the first identity, we notice that

$$R(p) \cdot [(\epsilon \cdot \Gamma_1) \circ (\Gamma_1 \cdot \eta)] = [R(p) \cdot (\epsilon \cdot \Gamma_1)] \circ [R(p) \cdot (\Gamma_1 \cdot \eta)] = (id_{R(p)}\Gamma_1) \circ (pe_0 \cdot \eta) = id_{R(p)\Gamma_1}$$

where the last identity follows from the fact that $pe_0 \cdot \eta = id_{pe_0} = id_{R(p)\Gamma_1}$. Similarly, we have

$$\hat{d}_1 \cdot [(\epsilon \cdot \Gamma_1) \circ (\Gamma_1 \cdot \eta)] = (\zeta^{-1} \cdot \hat{d}_1\Gamma_1) \circ (e_1 \cdot \eta) = (\zeta^{-1} \cdot e_1) \circ (\zeta \cdot e_1) = id_{\hat{d}_1\Gamma_1}$$

Therefore, $(\epsilon \cdot \Gamma_1) \circ (\Gamma_1 \cdot \eta) = id_{\Gamma_1}$. To prove the second identity, $(\Lambda_1 \cdot \epsilon) \circ (\eta \cdot \Lambda_1) = id_{\Lambda_1}$, we first prove the following lemma:

LEMMA 5.4. $\Gamma_1 \cdot \tau_0 \cdot \Lambda_1 = R(\mathfrak{a}) \cdot \lambda_p$

Proof. First we verify that the domain and codomain of these 2-cells match.

$$\begin{array}{ccccc}
 B/p & \xrightarrow{\Lambda_1} & E & \xrightarrow{\Gamma_1} & B/p \\
 & & \begin{array}{ccc}
 \xrightarrow{e_0} & E & \xrightarrow{i_E} \\
 & \tau_0 \Downarrow & \\
 & & E
 \end{array} & & \\
 & & \text{1} & &
 \end{array}$$

Indeed,

$$\Gamma_1 i_E e_0 \Lambda_1 = i(p) e_0 \Lambda_1 = i(p) \mathfrak{a} = R(\mathfrak{a}) i(R(p))$$

and as we observed earlier $\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a}) R(i(p))$. So, the domain and codomain of $\Gamma_1 \cdot \tau_0 \cdot \Lambda_1$ and $R(\mathfrak{a}) \cdot \lambda_p$ agree. The lemma follows from identities in below in conjunction with comma property of B/p for 2-cells.

$$\hat{d}_1 \cdot (\Gamma_1 \cdot \tau_0 \cdot \Lambda_1) = \phi_E \cdot \Lambda_1 = \mathfrak{a} \tau_1(p) = \widehat{\mathfrak{a} \hat{d}_1} \cdot \lambda_p = \hat{d}_1 \cdot R(\mathfrak{a}) \cdot \lambda_p$$

$$R(p) \cdot (\Gamma_1 \cdot \tau_0 \cdot \Lambda_1) = id_{pe_0} \cdot \lambda_1 = id = R^2(p) \cdot \lambda_p = R(p) R(\mathfrak{a}) \cdot \lambda_p$$

□

⁷Perhaps, it is illuminating to see what this unit look like in the case of $\mathcal{K} = \mathfrak{C}\mathfrak{a}\mathfrak{t}$. Indeed, for a morphism $f: e_0 \rightarrow e_1$ in E^{\rightarrow} , $\eta(f)$ is given as follows:

$$\begin{array}{ccccc}
 e_0 & \xrightarrow{\zeta_{e_0}(f)} & \mathfrak{a}\langle 1_{p(e_0)}, e_0 \rangle & \xrightarrow{\mathfrak{a}\Gamma_1\tau_0(f)} & \mathfrak{a}\langle p(f), e_1 \rangle \\
 \downarrow f & & & & \downarrow \Lambda_1\Gamma_1(f) \\
 e_1 & \xrightarrow{\zeta_{e_1}(f)} & & & \mathfrak{a}\langle 1_{p(e_1)}, e_1 \rangle
 \end{array}$$

Using lemma above we have,

$$e_0 \cdot [(\Lambda_1 \cdot \epsilon) \circ (\eta \cdot \Lambda_1)] = (\mathbf{a} \cdot \epsilon) \circ ((\mathbf{a} \Gamma_1 \tau_0) \circ (\zeta e_0)) \cdot \Lambda_1 = (\mathbf{a} \cdot R(\zeta^{-1})) \circ (\mathbf{a} R(\mathbf{a}) \cdot \lambda_p) \circ (\zeta \mathbf{a}) = (\zeta^{-1} \mathbf{a}) \circ (\zeta \mathbf{a}) = id_{e_0 \Lambda_1}$$

The penultimate equality comes from equality of pasting diagrams 5. Similarly, using the fact that $e_1 \Lambda_1 = ai(p) \hat{d}_1$, we get

$$e_1 \cdot [(\Lambda_1 \cdot \epsilon) \circ (\eta \cdot \Lambda_1)] = (\mathbf{a} i(p) \hat{d}_1 \cdot \epsilon) \circ (\zeta \cdot e_1 \Lambda_1) = (\mathbf{a} i(p) \zeta^{-1} \hat{d}_1) \circ (\zeta \cdot \mathbf{a} i(p) \hat{d}_1) = id_{e_1 \Lambda_1}$$

The last identity is by exchange law of horizontal-vertical composition of 2-cells. From these two equations we deduce the second adjunction identity.

Conversely, suppose we are given a Chevalley adjunction, that is to say a right adjunction Λ_1 of Γ_1 over B :

$$\begin{array}{ccc} \begin{array}{c} \eta \\ \curvearrowright \\ E \end{array} & \begin{array}{c} \xrightarrow{\Gamma_1} \\ \perp \\ \xleftarrow{\Lambda_1} \end{array} & \begin{array}{c} \epsilon \\ \curvearrowright \\ B/p \end{array} \\ & \begin{array}{c} \searrow pe_0 \\ \swarrow R(p) \end{array} & & \\ & & & B \end{array}$$

such that the counit ϵ is an isomorphism, $R(p) \Gamma_1 = pe_0$, $pe_0 \Lambda_1 = R(p)$, $R(p) \cdot \epsilon = id_{R(p)}$, and $pe_0 \cdot \eta = id_{pe_0}$. We define pseudo-algebra $\mathbf{a}: B/p \rightarrow E$ as composite $e_0 \Lambda_1$. Note that $p\mathbf{a} = pe_0 \Lambda_1 = R(p) \Gamma_1 \Lambda_1 = R(p)$, since the adjunction $\Gamma_1 \dashv \Lambda_1$ takes place in \mathcal{K}/B . We propose $e_1 \eta i_E$ for ζ . First we prove that $\eta \cdot i_E$ is invertible and thence ζ is invertible. Using $\tau_1 \cdot i_E = id$, we have $(i_E \hat{d}_1 \epsilon \cdot i(p)) \circ (\tau_1 \cdot \Lambda_1 \Gamma_1 i_E) \circ (\eta \cdot i_E) = id_{i_E}$. This is illustrated in the following pasting equality⁸:

$$\begin{array}{ccccc} & & & & 1 \\ & & & & \tau_1 \Downarrow \\ E \rightarrow & \xrightarrow{1} & E \rightarrow & \xrightarrow{e_1} & E \xrightarrow{i_E} E \rightarrow \\ \uparrow i_E & \searrow \Gamma_1 & \uparrow \Lambda_1 & \searrow \Gamma_1 & \uparrow i_E \\ E & \xrightarrow{i_p} & B/p \xrightarrow{1} & B/p \xrightarrow{\hat{d}_1} & E \\ & & & & \uparrow i_E \\ & & & & 1 \\ & & & & \uparrow i_E \\ & & & & id \Downarrow \\ E & \xrightarrow{1} & E & \xrightarrow{1} & E \end{array} = \begin{array}{ccc} E \rightarrow & \xrightarrow{1} & E \rightarrow \\ \uparrow i_E & & \uparrow i_E \\ E & \xrightarrow{1} & E \end{array}$$

Using lemma 5.1, we conclude that pasting diagrams shown below are equal and it follows that $(i_E \hat{d}_1 \cdot \epsilon \cdot i(p)) \circ (\tau_1 \cdot \Lambda_1 \Gamma_1 i_E)$ is also a right inverse, hence a 2-sided inverse of $\eta \cdot i_E$.

⁸This equality of course lies over B .

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 E & & E^{\rightarrow} & \xrightarrow{1} & E^{\rightarrow} & \xrightarrow{1} & E^{\rightarrow} \\
 \downarrow i(p) & \nearrow \Lambda_1 & \downarrow \Gamma_1 & \Downarrow \tau_1 & \downarrow \Gamma_1 & \Downarrow \eta & \\
 B/p & \xrightarrow{\epsilon} & B/p & \xrightarrow{1} & B/p & \xrightarrow{\Lambda_1} & E^{\rightarrow} \\
 & & \downarrow \tau(p) & & & & \\
 & & B/p & & & &
 \end{array} & = & \Lambda_1 i(p) \left(\begin{array}{c} E \\ \Downarrow \\ E^{\rightarrow} \end{array} \right) \Lambda_1 i(p)
 \end{array}$$

Whiskering with e_1 reveals inverse of ζ :

$$\zeta^{-1} = (e_1 i_E \hat{d}_1 \cdot \epsilon \cdot i(p)) \circ (e_1 \cdot \tau_1 \cdot \Lambda_1 \Gamma_1 i_E) = \hat{d}_1 \cdot \epsilon \cdot i(p)$$

since $e_1 \cdot \tau_1 = id_{e_1}$. Indeed, ζ^{-1} is the counit of composite adjunction in below:

$$\begin{array}{ccccc}
 E & \xrightarrow{i_E} & E^{\rightarrow} & \xrightarrow{\Gamma_1} & B/p & \xrightarrow{\hat{d}_1} & E \\
 \leftarrow e_0 & \perp & \leftarrow \Lambda_1 & \perp & \leftarrow i(p) & \perp & \\
 & & & & & &
 \end{array}$$

To finish the proof, by lemma 2.5 it suffices to prove that $\zeta^{-1} = \hat{d}_1 \cdot \epsilon \cdot i(p)$ satisfies pasting equality in (5), and moreover, $i(p) \dashv a$ with ζ and $R(\zeta^{-1}) \circ (R(\mathbf{a}) \cdot \lambda_p)$ as unit and counit of this adjunction respectively. \square

REMARK 5.5. Notice that we have proved ζ is invertible regardless of invertibility of ϵ . Also, obviously ϵ is identity (iso) then if and only if ζ is identity (iso) 2-cell.

EXAMPLE 5.6. We now return to prove our promise at the end of example 4.4. We would like to show that \tilde{f} , obtained as whiskering \hat{d}_1 with counit of $i(p) \dashv \mathbf{a}$, is indeed cartesian. Here, we appeal to the bijection $\text{Hom}_{B/p}(\Gamma_1(g), \langle f, e_1 \rangle) \cong \text{Hom}_{E^{\rightarrow}}(g, \Lambda_1 \langle f, e_1 \rangle)$ natural in $g: d_0 \rightarrow d_1$ in E^{\rightarrow} and $\langle f, e_1 \rangle$ in B/p . This bijection states that any diagram of the form

$$\begin{array}{ccccc}
 & & d_1 & \xrightarrow{k} & e_1 \\
 & & \downarrow p & & \downarrow p \\
 p(d_0) & \xrightarrow{p(g)} & p(d_1) & \xrightarrow{h_1} & b_1 \\
 \searrow h_0 & & & \nearrow h_1 & \\
 & & b_0 & \xrightarrow{f} & b_1
 \end{array}$$

where the square in base commutes and k lies above h_1 can be (uniquely) completed to the diagram below:

$$\begin{array}{ccccc}
 d_0 & \xrightarrow{g} & d_1 & & \\
 \downarrow \bar{h}_0 & \searrow & \downarrow & \searrow k & \\
 & \mathbf{a}(f, e_1) & \longrightarrow & \mathbf{a}(1_{b_1}, e_1) & \xrightarrow{\zeta^{-1}e_1} e_1 \\
 & \downarrow & & \downarrow & \downarrow h_1 \\
 p(d_0) & \xrightarrow{\dots} & p(d_1) & \xrightarrow{\dots} & b_1 \\
 \downarrow h_0 & & \downarrow & & \downarrow \\
 b_0 & \xrightarrow{f} & b_1 & \xrightarrow{=} & b_1
 \end{array}$$

Taking g to be identity we obtain the usual condition which expresses cartesian property of lift \tilde{f} . Also, one can easily show that unique morphism \bar{h}_0 over h_0 is calculated by the expression $(e_0 \Lambda_1(h_0, h_1, k)) \circ (\mathbf{a} \Gamma_1 \tau_0(g)) \circ (\zeta e_0(g))$.

EXAMPLE 5.7. Let $p: E \rightarrow B$ be a cloven Grothendieck fibration. Note that the data of a cloven Grothendieck fibration includes structure of a cleavage, that is a choice of cartesian lifts:

$$\rho_{a,b} : \prod_{\text{Hom}(a,b)} \prod_{e \in E_b} \sum_{e' \in E_a} \mathcal{C}art_E(e', e)$$

where $\mathcal{C}art_E(e', e)$ denotes the set of cartesian morphisms from e' to e .

We say p is *split* if for all pairs of objects a, b :

$$\text{snd } \rho_{a,c}(g \circ f, e) = \text{snd } \rho_{b,c}(g, e) \circ \text{snd } \rho_{a,b}(f, \text{fst } \rho_{b,c}(g, e))$$

and we say p is *normal* if for all objects e in E :

$$\text{snd } \rho_{pe,pe}(1_{pe}, e) = 1_e$$

We have the following correspondence:

$$\left\{ \begin{array}{c} \text{cleavages} \\ \text{of } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{pseudo-algebras} \\ (\mathbf{a}, \zeta, \theta) \text{ of } R \text{ at } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{right adjoints of } \Gamma_1 \\ \text{with isomorphism counit} \end{array} \right\}$$

It follows that any two cleavages of p are isomorphic in a unique way.

DEFINITION 5.8. For a category B , define 2-category $\mathbf{Fib}(B)$ of fibrations over B whose 0-cells are Grothendieck fibrations, whose 1-cells are fibred functors over B (i.e. those functors over B which preserve cartesian morphisms), and 2-cells are vertical natural transformations (i.e. transformations over B). Compositions are usual composition of functors and natural transformations.

REMARK 5.9. Example 4.4 can be encapsulated as follows: The forgetful 2-functor $U: \mathbf{Fib}(B) \rightarrow \mathcal{C}at/B$ is 2-monadic: the *free fibration* of a functor $p: E \rightarrow B$ is fibration $R(p): B/p \rightarrow B$; cleavage (aka fibration structure) on p is uniquely (in fact unique up to unique isomorphism) determined by a pseudo algebra structure for 2-monad $R = UF$. Strict algebra structures of R correspond to splitting fibration structures on p .

$$\begin{array}{c}
 \mathbf{Fib}(B) \\
 \left. \begin{array}{c} \uparrow \\ F \quad \dashv \\ \downarrow \\ \mathcal{C}at/B \end{array} \right\} U \\
 \mathcal{C}at/B \xleftarrow{R} R
 \end{array}$$

We also note that for a category B the domain functor $\text{cod}: B^{\rightarrow} \rightarrow B$ is the free Grothendieck fibration on identity functor $1: B \rightarrow B$; that is $\text{dom} = R(1)$. In more explanatory terms this fact states that

We also note that for a category B with pullbacks the codomain functor $\text{cod}: B^{\rightarrow} \rightarrow B$ is the free Grothendieck fibration *with existential quantifiers* on identity functor $1: B \rightarrow B$;

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