Surjection of locales

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Abstract

These are extended notes based on Steve Vickers's talk in Category Reading Group meeting on February 8, 2018. For writing them I also consulted (MacLane & Moerdijk 1992) and (Johnstone 1982). The main point of the notes is to show that the notion of surjection of locales differ from that of point-set topology. The embedding of locales are one-to-one on points, but there are surjection of locales which are not onto on points.

1 Points of locales

Definition 1.1. A L ***frame*** is a complete lattice in which finite meets distribute over arbitrary joins, that is

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

for any $a \in L$ and $S \subset L$.

Notice that in frame we have both arbitrary joins as well as arbitrary meets. However the meets are the not the "right" operations for purposes of geometric reasoning. This is reflected in how we define the notion of homomorphisms of frames.

Definition 1.2. A ***frame homomorphism*** from a frame F to frame L is a function which preserves finite meets and arbitrary joins.

Remark 1.3. The nullary cases of above definition tells us that $f(0) = 0$ and $f(1) = 1$ where 0 and 1 are the top elements of each frame resp.

Frames and frame homomorphism form a category $\mathcal{F}rm$. Note that the category $\mathcal{F}rm$ is not cartesian closed since hom objects are not frames; suppose F and L are frames, then $\mathcal{F}rm(F, L)$ is a poset which has only directed joins but not all joins. The order is given as

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follows: for $f, g\colon F \rightrightarrows L$, $f \le g$ if $f(x) \le g(x)$ in L for any $x \in F$. Viewing F, L as categories and f, g as functors, this order is just a natural transformation between f and g. To prove that $\mathcal{F}rm(F, L)$ has directed joins, take a directed family of frame morphisms $\{f_i\}_{i \in I}$. Define $\bigvee^{\uparrow} f_i(x) = \bigvee f_i(x)$. To prove that $\bigvee^{\uparrow} f_i$ preserves meets we need the condition that $\{f_i\}_{i \in I}$ is directed.

Example 1.4. Every topological space E has a frame of open subsets denoted by $O(E)$. Every continuous map of spaces, say $p\colon E \to B$ gives rise to a map of corresponding frames of opens in the opposite direction, namely inverse image map $p^*\colon \mathcal{O}(B) \to \mathcal{O}(E)$. By AFT p^* has a right adjoint $p_* : \mathcal{O}(E) \to \mathcal{O}(B)$ given by

$$p_*(V) = \bigvee_{p^*U \le V} U$$

Notice that $p_*(E) = B$ as expected since right adjoints preserve limits. On the other hand, if p is not surjective then $p_*(\emptyset)$ may not be $\emptyset$. The unit of adjunction above says $U \subset p_* p^* U$ for any open U of B, and the counit says that $p^* p_* V \subset V$ for any open V of E. Also if p is embedding then what $p_* V$ computes is the biggest open subset W of B for which $W \cap E = V$.

Definition 1.5. The category $\mathcal{L}oc$ of ***locales*** is defined as the opposite of category of frames. A locale is a frame thought of as an object of $\mathcal{F}rm^{\mathrm{op}}$. Locale maps are frame homomorphism backwards. To preserve this distinction in our mind we choose letters $X, Y, Z, \ldots$ to denote locales and $\mathcal{O}(X), \mathcal{O}(Y), \mathcal{O}(Z), \ldots$ for the corresponding frames in the opposite category. For a map $f\colon X \to Y$ of locales we denote the corresponding frame homomorphism by $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. Of course as discussed above by AFT, f^* has a right adjoint $f_*\colon \mathcal{O}(X) \to \mathcal{O}(Y)$.

In general, a locale can be regarded as a sort of generalized topological space in which one may speak of open sets but one does not generally have a sufficient supply of points.

Definition 1.6. The terminal object in category $\mathcal{L}oc$ is called terminal locale and will be denoted by 1. Note that the corresponding frame is the initial frame $\mathcal{O}(1) = \{0 \le 1\}$ which has only two elements got by empty meet and empty join. The initial locale has terminal frame of opens $\{0 = 1\}$

Definition 1.7. A ***filter*** F of a lattice L is an upward closed subset $F \subset L$ such that it contains the top element $\top$ and meet of any two of its elements. We can use join and meet operations instead of order to present axioms of filter:

$$\vdash \top \in F$$

$$a \in F, b \in F \vdash a \wedge b \in F$$

$$a \in F, b \in B \vdash a \vee b \in F$$

A ***prime filter*** is a filter satisfying following axioms in addition:

$$\bigvee x_i \in F \vdash \exists i \in I \,\textbf{.}\, x_i \in F$$

for any finite indexing set I. Taking I to be empty set, we deduce that bottom element of lattice cannot be in the prime filter. Note that only nullary and binary cases are equivalent to the axiom above. A ***complete prime filter*** is a filter for which the above axiom is true for any indexing set I not necessarily finite.

Definition 1.8. A ***point*** of a locale X is a global section of X. So,

$$\left\{ \text{ points of } X \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{frame morphisms} \\ \mathcal{O}(X) \to \mathcal{O}(1) \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{complete prime} \\ \text{filters of } \mathcal{O}(X) \end{array} \right\}$$

Example 1.9. Consider a locale $\mathbb{S}$ whose frame is given by $\mathcal{O}(\mathbb{S}) = \{0 \le a \le 1\}$. This locale is called Sierpiński locale. Note that $\mathbb{S}$ has only two points $\bot, \top$; where $\bot^* a = 0$ and $\top^* a = 1$. Point $\bot$ is closed and point $\top$ is open. Also, Sierpiński locale is a classifier for frame of opens of a locale; that is for any locale X,

$$\mathcal{O}(X) \cong \mathcal{L}oc(X, \mathbb{S})$$

is a bijection of sets. We can think of opens of a locale (i.e. elements of corresponding frame) as proposition symbols of a propositional theory which presents that locale and points as a model of the theory. Now suppose we have a point $x\colon 1 \to X$ and an open $U\colon X \to S$ of a locale X. Then we write $x \models U$ whenever $U \circ x = \top$, or equivalently $x^* U = 1$.

Definition 1.10. A locale X is said to be ***spatial*** if it has enough points, that is if elements of frame $\mathcal{O}(X)$ (i.e. propositions) can be distinguished by points (i.e. models) of X. More specifically, for any two distinct opens $U, V \in \mathcal{O}(X)$, there is a point $x\colon 1 \to X$ such that $U \circ x \neq V \circ x$. Put differently, for any two distinct propositions U, V, there is a model x of X such that x satisfies one proposition but not the other.

As we observed For every topological space E, the collection $\mathcal{O}(E)$ of open subsets of E forms a frame. So E can be regarded as a locale which we denote by $L(E)$. Note that $\mathcal{O}(L(E)) = \mathcal{O}(E)$. Conversely, if X is a locale, then there is a natural topology on the set $pt(X)$ of points of X which extracts a topological space from X. The opens are defined to be

$$\begin{aligned} pt(U) &= \{x \in pt(X) : x \models U\} \\ &= \{\text{complete prime filters } \mathcal{F} \subset \mathcal{O}(X) : U \in \mathcal{F}\} \end{aligned}$$

Notice that by definition of complete prime filter we have

$$pt(U \wedge V) = pt(U) \cap pt(V)$$
$$pt(\bigvee U_i) = \bigvee pt(U_i)$$

Any maps of locales $f\colon X \to Y$ induces a function which takes points of X to points of Y:

$$pt(f)\colon pt(X) \to pt(Y)$$

This is achieved simply by post composition with f. Note that $pt(f)$ is indeed a continuous map of topological spaces:

$$\begin{aligned} x \in pt(f)^{-1}(pt(V)) &\iff f \circ x \models V \\ &\iff V \circ f \circ x = \top \\ &\iff x \in pt(f^* V) \end{aligned}$$

Therefore, $pt(f)^{-1}(pt(V)) = pt(f^*V)$ is open in $pt(X)$ which proves $pt(f)$ is continuous map of topological spaces. These two constructions are adjoint to one another:

$$\mathcal{T}op \underset{pt}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{L}oc$$

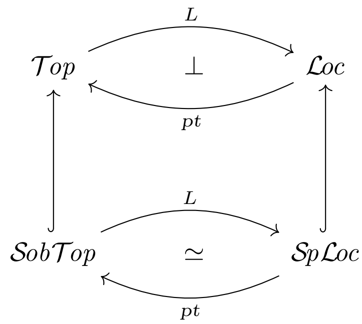
For a topological space E, the unit of this adjunction $\eta_E \colon E \to pt(L(E))$ takes a point $e \in E$ to the complete prime filter of open neighbourhoods of e. More logically still, $\eta_E(e) = \{x \colon 1 \to L(E) \mid x \models V$ for all open V contaning $e\}$. We observe that η_E is indeed an open map of topological spaces, since $\eta_E()$ Note that for a locale X, the counit of adjunction $\epsilon_X \colon L(pt(X)) \to X$ is a locale map given by $\epsilon_X^*(U) = pt(U)$ for any open $U \in \mathcal{O}(X)$. Note that ϵ_X^* is always surjective. In good cases these adjoint functors are actually inverse equivalences. More precisely, the adjunction gives rise to an equivalence between the category of spatial locales and the category of sober topological spaces. This is famous Stone duality.

Remark 1.11. A locale X is spatial iff the functor pt is injective on objects, that is, for any $U, V \in \mathcal{O}(X)$, $pt(U) = pt(V)$ implies $U = V$. From this observation it follows that X is spatial iff counit $\epsilon_X \colon L(pt(X)) \to X$ is an isomorphism of locales, hence X is isomorphic to the locale of a topological space, namely the space of its points.

Definition 1.12. A ***sober*** space is a topological space X such that every irreducible closed subset of X is the closure of exactly one point of X: that is, this closed subset has a *unique* generic point. Any sober space is T_0. Every affine scheme (i.e. an scheme homeomorphic to $Spec(R)$ for a commutative ring R with the Zariski topology) is a compact sober space. More generally, the underlying topological space of any scheme is a sober space.

Proposition 1.13. A topological space E is sober iff the unit $\eta_E \colon E \to pt(L(E))$ is a homeomorphism. Thus a sober space E can be recovered as the space of points of its locale.

Proof. For the proof see proposition IX.3.2. of (MacLane & Moerdijk 1992). □



Example 1.14. Here we give an example of a locale with no points. Let X be a locale whose frame of opens $O(X) = B$ is a complete Boolean algebra. Points of locale X which are in bijection with complete prime filters of B. Now, any prime filter P of a B is in fact an ultrafilter: simply because $a \vee \neg a = 1 \in P$ so either $a \in P$ or $\neg a \in P$. For a completely prime filter P of B, since B is complete we can form $\bigwedge P$ and show that it is indeed an atom of B. For if $x \in B$ with $x < \bigwedge P$ then x is not in P. Therefore $\neg x \in P$, and $x < \bigwedge P < \neg x$. This implies $x = x \wedge \neg x = \perp$.

Conversely, if a is an atom of Boolean algebra B, then the principal filter $B \uparrow a$ is an ultrafilter and hence a compeltely prime filter, and thus a point of X. Upward closedness of P implies that $\uparrow \bigwedge P \cong P$. This completes the proof of one-to-one correspondence of complete prime filters of B with atoms of B.

In summary, for an atomless Boolean algebra B, X is a locale without any points. For an example of atomless Boolean algebra consider the Boolean algebra of *regular*[1] open sets of affine n-dimensional space $\mathbb{R}^n$. The negation is got by taking interior of complement of each open. Thus, a regular open is an open set whose closure's interior is equal to itself. The intuition is that regular opens do not have holes. One readily checks that this Boolean algebra does not have any atoms.

2 Image factorization for maps of locales

In this section we will give a review of image factorizations for maps of locales. In order to obtain it we simply give image factorizations of morphisms of corresponding frames. There will be some new terminologies such as closure, coclosure operators, nucleus, etc about posets that are probably more familiar as monads, comonads, lex functors, etc to category theorists. Considering every poset as a category, you can consult left column in the table below to recollect the definition of new terms on the left column:

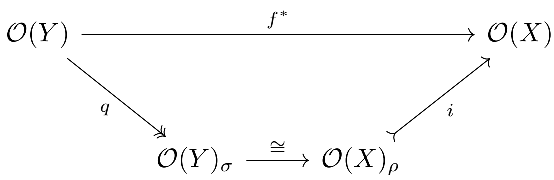
[1] In general, an element of a Heyting algebra is said to be regular whenever $\neg\neg x = x$, where $\neg x = x \Rightarrow \perp$. Thus a Boolean algebra is precisely a Heyting algebra in which every element is regular.

Posets	Categories
A has finite meets	A is cartesian (i.e. has finite products)
A has joins	A has small coproducts
$T\colon A \to A$ is a (co)closure operator	$T\colon A \to A$ is a (co)monad
Fixed-points of T	$\mathbb{A}\mathrm{lg}(T)$ / $\mathbb{C}\mathrm{oalg}(\mathrm{T})$
$T : A \to A$ is a nucleus	$T : A \to A$ is a lex monad
$N(A)$: frame of nucleus on A	$\mathbb{L}\mathrm{ex}\mathbb{M}\mathrm{nd}(A, A)$: functor category of lex monads on A

Suppose $f\colon X \to Y$ is a map of locales and $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is the corresponding frame homomorphism. As we observed in the previous section, f^* has a right adjoint f_* which makes $\sigma := f_* f^*$ a closure operator on $\mathcal{O}(Y)$ and $\rho := f^* f_*$ a coclosure operator on $\mathcal{O}(X)$. Indeed, since both f^* and f_* preserve finite meets, both σ is a nucleus and ρ is a conucleus. Define

$$\mathcal{O}(Y)_\sigma = \{V \in \mathcal{O}(Y) : V = \sigma V\}, \quad \mathcal{O}(X)_\rho = \{U \in \mathcal{O}(X) : \rho U = U\}$$

Now, we can factorize f^* as follows:

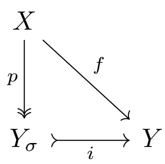


where $q(V) := \sigma V$. Notice that q is surjective and hence an epimorphism (in fact, a regular epimorphism) of frames. Similarly, i is inclusion and hence it is mono. The isomorphism in the above diagram is given by f^* restricted to $\mathcal{O}(Y)_\sigma$.

Remark 2.1. From factorization above we learn that if f^* is one-to-one then q must be identity and hence for every $V \in \mathcal{O}(Y)$, $V = \sigma V = f_* f^* V$. So, f^* is a section of f_*, hence f_* is an epimorphism of frames. Similarly, if f^* is onto then i must be identity, and as a result f_* has a left inverse, and thus f_* is a monomorphism of frames.

Definition 2.2. Suppose $f\colon X \to Y$ is a map of locales. We call f a ***surjection*** if f^* is one-to-one and an ***embedding*** if f^* is onto.

Proposition 2.3. Any map $f\colon X \to Y$ of locales can be factorized as a surjection followed by an embedding.



where $\mathcal{O}(Y_\sigma) = \mathcal{O}(Y)_\sigma$. Moreover, this factorization is unique up to unique isomorphism.

Proof. See theorems IX.4.4 and IX.4.6 of (MacLane & Moerdijk 1992). ☐

Example 2.4. Let X be a locale. Let $U \in \mathcal{O}(X)$.[2] Clearly, $\downarrow U$ is a frame, and distributivity law of frame $\mathcal{O}(X)$ ensures that the function $- \wedge U : \mathcal{O}(X) \to \downarrow U$ is a surjective frame homomorphism. From this we obtain nucleus as $\sigma = U \Rightarrow (-)$. We denote the sublocale X_σ by U. The sublocales obtained this way are called *open sublocales*.

3 A surjection of locales which is not onto on points

Definition 3.1. Let P be a poset. An ***ideal*** I of P is a subset of P such that

1. I is down-closed.
2. Any finite subset of I has an upper bound in I.

The two important cases of second axiom, nullary and binary cases imply that I is inhabited and secondly, for any two elements $x, y \in I$ there is a third element $z \in I$ such that $x, y \leq z$.

We write $\mathcal{I}dl(P)$ for the set of ideals of poset P. Notice that $\mathcal{I}dl(P)$ is not a lattice in general. However it has a (subset) order and in fact with this order it is directed complete, that is any directed set of ideals of P, say $\{I_\lambda : \lambda \in \Lambda\}$ has a suprema, namely $\bigcup_\lambda I_\lambda$ which is fairly easy to check it is an ideal.

We can equip $\mathcal{I}dl(P)$ with Scott topology; declare a subset $\mathcal{U} \subset \mathcal{I}dl(P)$ open if $\mathcal{U}$ is up-closed and it is inaccessible by directed joins. The latter condition means that if $\bigvee^{\uparrow} I_\lambda \in \mathcal{U}$, then there is at least one $\lambda \in \Lambda$ such that $I_\lambda \in \mathcal{U}$. We have to check that the collection of Scott opens do indeed form a topology. Here, we only show that they are closed under finite intersection, and we leave the rest to the reader. Suppose $\mathcal{U}_i$, for $1 \leq i \leq n$, are open. Obviously $\bigcap \mathcal{U}_i$ is up-closed. Let $\bigvee^{\uparrow} I_\lambda \in \bigcap \mathcal{U}_i$. Then for each i, $\bigvee^{\uparrow} I_\lambda \in \bigcap \mathcal{U}_i$ and we can find some $\lambda_i \in \Lambda$ such that $I_{\lambda_i} \in \mathcal{U}_i$. Because of directedness, we can find some $\gamma \in \Lambda$ such that $I_{\lambda_i} I_\gamma$ for each $1 \leq i \leq n$. Because of up-closedness of each $\mathcal{U}_i$, we have $I_\gamma \in \mathcal{U}_i$ for each i. Therefore, $I_\gamma \in \bigcap \mathcal{U}_i$.

Let P be a meet-semilattice equipped with a coverage $\mathbb{T}$. For instance if P is a distributive lattice, we can take $\mathbb{T}$ to defined as follows: A finite $S \subset \downarrow p$ is a cover for $p \in P$ whenever $\bigvee S = a$. Distributivity law guarantees that condition of coverage is satisfied. A subterminal sheaf on the site $(P, \mathbb{T})$ is just an ideal of P in the sense of definition (3.1). So, we have

$$\mathcal{O}(Sh(P, \mathbb{T})) \cong \mathcal{I}dl(P)$$

Proposition 3.2. If P is a distributive lattice then $\mathcal{I}dl(P)$ is a frame.

Proof. The lattice of opens of every topos is a complete Heyting algebra, hence a frame. ☐

[2]The frame of every locale is non-empty even for the initial locale this is true.

For more on $(P, \mathbb{T})$-ideals we refer the reader to (Johnstone 1982).

To the Scott topology described above we can associate a specialization order on $\mathcal{I}dl(P)$ as follows: for two ideals I, J, we define $I \sqsubseteq J$ whenever for all Scott opens $\mathcal{U}$ if $I \in \mathcal{U}$ then $J \in \mathcal{U}$. One can easily prove that specialization order in this case is same thing as subset order. Also, a map $f\colon \mathcal{I}dl(P) \to \mathcal{I}dl(Q)$ is Scott continuous iff f preserves directed joins.

To describe Scott opens more concretely, consider for each $x \in P$, the principal ideal $P \downarrow x$. Every ideal I can be obtained as directed joins of principal ideals: $I = \bigvee\{P \downarrow x : x \in I\}$. Notice that if $\mathcal{U}$ is Scott open then

$$I \in \mathcal{U} \iff \exists x \in I \,.\, P \downarrow x \in \mathcal{U}$$

With this observation to every Scott open $\mathcal{U}$ of $\mathcal{I}dl(P)$ we can uniquely associate an up-closed subset of P, namely $U = \{x : P \downarrow x \in \mathcal{U}\}$. Conversely, to an up-closed subset $U \subset P$ we assign the Scott open $\mathcal{U} := \{\text{ideal } I : P \downarrow x \subset I \text{ for some } x \in U\}$. In this way there is a bijection between Scott opens of $\mathcal{I}dl(P)$ and up-closed subsets of poset P.

Now, make poset P into a locale with discrete topology so that $\mathcal{O}(P) = \mathbb{P}(P)$ is the power set of P. There is a one-to-one map of frames $\mathcal{O}(\mathcal{I}dl(P)) \hookrightarrow \mathcal{O}(P)$ which makes the corresponding locale map $P \twoheadrightarrow \mathcal{I}dl(P)$ is a surjection of locales. However, it is not onto at the level of points: Since P is discrete its points are its elements, and points of $\mathcal{I}dl(P)$ are ideals of P. The locale surjection takes a point x to $P{\downarrow}x$, however not all ideals are principal ideals. For an easy counter example consider the linear poset $\omega := \{0 \leq 1 \leq 2 \leq \ldots\}$, where ω itself is an ideal but not a principal one.

Remark 3.3. A more complicated surjection of locales which is not onto on points is the surjection from the locale of Cauchy real numbers to the locales of Dedekind reals.

4 Sublocales of pointful loacles may be pointless

Suppose L is a locale. Consider double negation nucleus $j\colon \mathcal{O}(L) \to \mathcal{O}(L)$ defined as $j(U) = \neg\neg U$. It induces a sublocale $u\colon L_j \to X$ for which $\mathcal{O}(L_j) = \mathbb{A}\mathrm{lg}(j) = \{U \in \mathcal{O}(X) : \neg\neg U = U\}$. We claim if X is a sober topological space without any isolated points then $L(X)_j$ does not have any points. To see this, suppose $p\colon 1 \to X_j$ is a point of locale X_j. The composition $up\colon 1 \to X$ is a point of $L(X)$, the locale of X. Since X is sober up corresponds to a point $x \in X$. Now, $X - \{x\}$ is open and $\neg(X - \{x\}) = \emptyset$, hence $\neg\neg(X - \{x\}) = X$. But since $x \notin X - \{x\}$, we have

$$0 = (up)^*(X - \{x\}) = p^* u^*(X - \{x\}) = p^* j(X - \{x\}) = p^*(X) = 1$$

a clear falsehood. Hence, point p does not exists.

This shows that the embedding $X_{\neg\neg}$ for a (non-empty) sober topological space X without any isolated points is not one-to-one on points, i.e. the map $pt(u)\colon pt(X_{\neg\neg}) \to pt(X)$ is not an injective map of sets.

In fact, for a topological space X, we can construct the largest sublocale of $L(X)$ which is pointless. For this we use the fact that every locale can be presented by a poset equipped with a coverage (aka covering system.) See (MacLane & Moerdijk 1992), exercise IX.5 on page 524. Consider the following coverage on frame $\mathcal{O}(X)$:

$$\{U_i\} \text{ covers } U \text{ iff } U - \bigcup U_i \text{ is finite.}$$

Write $K(X)$ for the locale presented by $\mathcal{O}(X)$ and this coverage. Then $K(X)$ is the largest sublocale of X which does not have any points.

Remark 4.1. Embedding of locales are monomorphims in category $\mathcal{L}oc$ of locales. IF $X \hookrightarrow Y$ is an embedding of locales, the function $pt(f) \colon pt(X) \to pt(Y)$ which is defined by post composition will be one-to-one.

References

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