What Are Derived Stacks?

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In the following paragraphs of this section, I will give a brief summary of my understanding of state of the art of study of stacks and higher categorical geometry.

1 Stacks

Very roughly, theory of stacks provides a mathematical formalism to glue together compatible geometrical objects which are varying over a given base space (such as vector bundles on topological spaces) where the gluing is subject to the constraints of topology of the base space. In the modern setting, à la Grothendieck, base space is taken in a more liberal sense to be groupoid (or a category) and still more generally an infinity groupoid where we include information of higher homotopies. In this setting, the fibred space is replaced by fibred category, and the data of topology of base space is replaced by a more general and elegant concept of Grothendieck coverage (Artin, 1962) (Artin, Grothendieck, & Verdier, 1972). With regard to a coverage, it makes sense to talk about gluing of objects over the base. A ***stack*** is then a fibred category in which not only can we glue compatible objects together but also the gluing is unique up to isomorphism in the total space. In this picture, one can view stacks as generalization of sheaves to the situations where one has to replace discrete fibres with groupoid fibres to include essential geometric data of mapping spaces of fibred spaces. A basic example is the classifying stack $\mathbb{B}G$ of a topological group G (or smooth affine group scheme). In this example, each fibre is the groupoid of principal G-bundles of some space (resp. scheme).

2 Higher stacks

For a moduli problem, stacks naturally appear as soon as objects must be classified up to isomorphism. However, we need higher stacks if objects must be classified up to a notion of equivalence, weaker than that of isomorphism. Example of this are sheaves of abelian groups up to quasi-isomorphisms and topological spaces up to weak homotopy equivalences. In order to coherently formalize this idea one is naturally lead to the structures of ∞-groupoid enriched categories, i.e. $(\infty, 1)$-categories.

3 Model categories and $(\infty, 1)$-categories

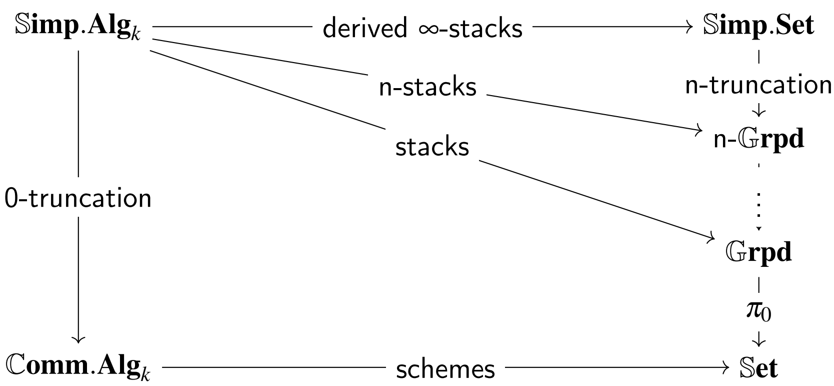
Model categories are strict forms of $(\infty, 1)$-categories, and model category theory is a strict form of the theory of $(\infty, 1)$-categories. See (Toën & Vezzosi, 2005). Although model categories are superior when it comes to explicit computation, $(\infty, 1)$-categories (e.g. Segal type categories) are better-suited for general functorial constructions, and many of the structures in category theory can be generalized to the setting of $(\infty, 1)$-categories. Using ideas both from model categories and technology of Segal categories as a good model of $(\infty, 1)$-categories, (Toën & Vezzosi, 2005) defines a general and precise notion of higher stacks based on several previous works, particularly (Dugger, Hollander, & Isaksen, 2004), (Simpson, 2004), and (Jardine, 2001).

4 Derived algebraic geometry

In local algebraic geometry, the fundamental object is the commutative ring; by using sheaf methods and glueing them together one forms the basic objects of global algebraic geometry. The field of derived algebraic geometry is to broaden the horizon by passing from commutative ring to commutative ring spectrum and transferring the constructions of local and global algebraic geometry to the setting of stable homotopy theory. It has already achieved important and interesting applications in homotopy theory, theory of manifolds and modular forms, and string theory in mathematical physics.

5 Derived stacks

Whereas higher stacks are defined on a usual 1-categorical site, derived stacks are defined on some $(\infty, 1)$-site. Having the schemes as functors point of view, the relationship between schemes, stacks, higher stacks, and derived higher stacks is best captured in the following diagram of categories and functors:



There are concrete advantages for working with derived stacks: passing to derived geometry makes colimits behave well in cohomology. Similarly for limits; for example, as shown by J. Lurie in his talk (Lurie, 2016) the intersection pairing of non-transversal smooth manifolds comes out correctly when regarding them as derived smooth manifolds instead of manifolds.

As one sees in the above diagram, one can regard study of stacks as a lift and refinement of of study of topological spaces[1]. This observation is justified, particularly from the point of view of application and computation, only if good old topological constructions and theories such as quotient spaces, fibrations, classifying spaces, homotopy, homology and cohomology, and K-theory of spaces among other things would nicely lift to the case of stacks and higher stacks. See for instance (Noohi, 2005), (Noohi, 2013).

6 Stacks in foundation of mathematics

Recently, in foundation of mathematics, stacks have appeared in (Coquand, Mannaa, & Ruch, 2017) as semantics of dependent type theory. In this model, types are interpreted as stacks, generalising the groupoid model of type theory. Establishing existence of such model, they prove that axiom of countable choice cannot be proved in dependent type theory with one univalent universe and propositional truncation. Also, in (Shulman, 2010), stacks are used to give an extension of usual internal logic of toposes to the case where one additionally has the unbounded quantification ranging over the class of objects of the topos, e.g. all groups,topological spaces, etc. internal to a topos.

7 Advantage of working with stacks

As discussed in section 1.3, one essential advantage of working with stacks and derived stacks is to make possible certain geometric constructions which do not exist in usual categories of spaces. More radically, we can expand our definition of space to have stacks as its primitive objects. One can then study formal properties of 2-category of stacks. One conjecture is to investigate whether this 2-category is indeed a 2-topos in the sense defined in (Weber, 2007). Furthermore, in the same spirit one research direction is to investigate formal properties and structures of higher stacks. A fundamental result here is that for any site $(\mathscr{C}, \mathbb{J})$, ∞-stacks over it form a Segal category, that is a model of $(\infty, 1)$-categories. Many of questions about structures of stacks can be asked in this $(\infty, 1)$-category of higher stacks, particularly what ∞-fibrations look like in here.

[1]which is historically very rich and one of substantial branches of mathematics.

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