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EPISTEMOLOGY, ONTOLOGY AND THE CONTINUUM

Standard wisdom has it that mathematical progress has eclipsed Kant's view of mathematics on three fronts: intuition, infinity and the continuum. Not surprisingly, these very areas define Brouwer's own relation to Kant, for Brouwer attempted to recreate the Kantian picture of the continuum by updating Kant's notions of infinity and intuition in a set theoretic context. I will show that when we look carefully at how Brouwer does this, we will find a certain internal tension (a"disequilibrium") between his epistemology of intuition and the ontology of infinite objects that he must adopt. However, I will also show that when we search the corresponding Kantian notions for that same disequilibrium, then — despite first impressions — we will find equilibrium instead. Because of this, I will suggest at the end that the basic components of Kant's eighteenth century view provide a foundation for important parts of classical rather than intuitionistic mathematics. This, in turn, will lead to a reassessment of Brouwer's Kantianism, and the way that mathematics has progressed from Kant's time.

The paper has four parts: First I will very quickly sketch the modern critique of Kant, and show how Brouwer recreated the Kantian continuum within set theory. I will also point out a flaw in Brouwer's early work, a flaw that he eventually noticed and corrected. In part II, I shall demonstrate that this flaw and its correction are special cases of a generalization that I have developed of the Hangman paradox. I will also show that this tool gives a precise characterization of Brouwer's mature mathematical program and the sense in which it is in disequilibrium. Then, in the third part I will argue that, in spite of the fact that modern intuitionism can interpret Kantian themes and texts, and in spite of Brouwer's own assessments, when we look more carefully we will find that Kant had an entirely different notion of intuition. This notion reflects influences as diverse as Leibniz and Euler, avoids the Brouwerian disequilibrium, and indeed moves closer to the classical than to the intuitionistic point of view about infinity and the continuum. Finally, in a brief concluding section, I will speculate that the Kantian equilibrium may teach quite a profound lesson about the classical continuum. I will also use this case study to question that "standard" conception of mathematical progress and to refine Professor Cellucci's distinction between the open and closed "world views" and his application of that distinction.

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KANT, BROUWER, AND THE CONTINUUM

Kant and the Standard Picture

Let me begin by summarizing our standard view about the ways in which modern set-theoretic mathematics has progressed from the eighteenth century. This view stems from criticisms raised by Bertrand Russell around the turn of the twentieth century. It has been widely accepted ever since.[1]

1. *Intuition.* The first difference is that mathematics now has an abstract character and is liberated from the connection with empirical intuition that characterized the eighteenth century. Russell argued vigorously that our new, powerful formal languages allow us to describe a vast range of abstractions and to denote objects which we may not even hope ever to perceive or construct.
2. *Infinity.* Certainly one of the greatest dividends of our new formal rigor and abstraction, according to Russell, is the modern attitude towards mathematical infinity. With Cantor's set theory mathematicians learned how to deal with infinity, and with modern notation we can rigorously express our new infinitary objects and operations. This modern embrace of the infinite replaces the abhorrence (or at best an ambivalence) that characterized earlier attitudes.
3. *Continuum.* Finally, the third difference is that set theory together with our new rigorous approach to infinity have engendered a profound change in the theory of the continuum. The early theory (whose origin is really in Aristotle) viewed the continuum as thick or "viscous:" its parts adhere to one another and cannot be separated without introducing a new gap. That is what distinguished continuity from mere infinite divisibility. In particular, the continuum was taken to be an independent unity, and it was not viewed as an aggregate of previously given and independently existing points.

Today, by contrast, we view the continuum set theoretically. **R** is the set of cauchy sequences of rational numbers, and we use methods of cardinality and topological techniques to distinguish continuity from infinite divisibility. Viscosity, in particular, has been replaced by the topological notion of connectedness: a space x is connected if x is not the disjoint union of two non-empty open sets. But this is a "thin" notion: you can neatly separate the space of real numbers by removing any single element. With the axiom of choice, **R** is even well-orderable.

There is good reason to take Kant's philosophy as exemplifying the eighteenth century view on each of these issues: intuition, an apparent ambivalence towards infinity, and continuity. Mathematics, Kant tells us, studies the forms of empirical objects, and so takes its cues from empirical perception. It would have no "objective validity" otherwise (A329/B299).

On the second point, his attitude to infinity seems ambivalent at best. Think of the confused treatment of infinity in the "Antinomies." Though the world continues without end, says Kant, nevertheless it is not infinite (A519/B547ff). And think of the explicit conflict between this denial that the world is spatially infinite with the claim in the "Aesthetic" that space *is* infinite (A25/B39).

Finally, on the the issue of the continuum: space (and indeed all continua) according to Kant are clearly viscous and not reducible to points. Indeed he describes continuity in terms of "flowing:"

> The property of magnitudes by which no part of them is the smallest possible, that is, by which no part is simple, is called their continuity. Space and time are *quanta continua,* because no part of them can be given save ... in such fashion that this part is itself again a space or a time. ... Points and instants are only limits, that is, mere position which limit space and time. But ... out of mere positions, viewed as constituents capable of being given prior to space or time, neither space nor time can be constructed. Such magnitudes may also be called *flowing*, since the synthesis of productive imagination involved in their production is a progression in time, and the continuity of time is ordinarily designated by the term flowing or flowing away. (A169-70/B211-12)

Brouwer on the Continuum

Now Brouwer is clearly in the modern camp with his whole-hearted reduction of analysis to set theory.[2] For Brouwer **R** is indeed a space of cauchy-sequences and real valued functions are defined over that space. Yet he still holds the eighteenth century view that the continuum is a viscous manifold. He makes this precise by means of his notorious uniform continuity theorem. Following is a simplified version of the theorem.

Any discontinuous real-valued function is not total. The main idea of the proof is quite simple: Suppose that $f{:}\mathbf{R}\rightarrow \mathbf{R}$ is discontinuous. Suppose in particular that $f(x) = 2$ for $x < 1$ and $f(x) = 3$ for $x \geq 1$. Brouwer then defines a point, $\xi \in \mathbf{R}$, which is so "indeterminate" that we cannot decide whether $\xi < 1$ or $\xi = 1$. $f(\xi)$ is uncalculable. And thus by intuitionistic standards, f will be undefined at ξ.

This theorem is not an isolated part of Brouwer's intuitionism. Indeed, the view of the continuum that it represents is at the heart of his work in all intuitionistic analysis and topology (Brouwer 1919).

Two corollaries follow quickly:

1. [0,2] (indeed all of **R**) is uncountable. This is because any total function attempting to enumerate the reals in that interval (or in **R**) would be similarly undefined at ξ.
2. **R** cannot be separated into any pair of disjoint non-empty subsets [i.e., there is no pair of non-empty sets (A, B) such that $\mathbf{R} = (A \cup B)$ and $(A \cap B) = \varnothing$].

The reason for this is that the characteristic functions f_A and f_B would have to be discontinuous. Thus, for instance, we can again use ξ to refute an alleged separation of **R** into $A = \{x : x < 1\}$ and $B = \{x : x \geq 1\}$. This is a stronger topological property than connectivity. It says that we cannot divide the real line into a pair of disconnected subsets by simply removing a single point. It precisely captures the intuitive notion of viscosity.

Corollary (1) is the heart of modern set theory. It gives Brouwer an alternative proof of the most basic theorem in our set theoretic picture of the continuum. But Corollary (2) is the heart of viscosity. And so with the introduction of "indeterminate reals" Brouwer has managed to bridge the two conceptions of the continuum. He has expressed the older, Kantian, viscous continuum in the idiom of modern set theory.

Brouwer and Intuition

How does he do this? Let's look at his definition of ξ. It is an early version of his notion of a "choice sequence," the notion that captures Brouwer's central doctrine about intuition. Indeed, Brouwer still clings to the Kantian dependence of mathematics on the form of intuition. He has, he says, rejected Kant's claims about the *a priority* of space. But he builds mathematics on the *a priori* intuition of time.[3] The notion of choice sequence captures one of the central features of temporal intuition: the sense of an open future. Following is a simplified sketch of his definition of ξ adapted from (Brouwer 1927):

> We construct the decimal expansion of ξ $(0.\ d_1\ d_2\ d_3,\ldots,\ d_n,\ldots)$ by means of an unlimited sequence of choices such that (a) we temporarily (*einstweilen)* choose for every k that we have already considered $d_1 = d_2 = \ldots = d_k = 9$. But (b) we reserve the right (*die Freiheit vorbehalten*) to determine at any time after the first, second, ... $(j\text{-}1)^{st}$, and j^{th} choice have been made the choice of all further elements [i.e., $(j+1)^{st}$, $(j+2)^{nd}$, and so on] in such a way that either (i) $d_j = 9$ or (ii) $d_j < 9$.

It is clear what Brouwer wants: he wants each digit in the decimal expansion of ξ to be chosen so that $\xi < 1$ or $\xi = 1$ always remain undecided (i.e., we cannot deduce from what we know àbout ξ exactly where it falls on the real line). And it is clear why he wants that: if ξ were ever to "settle down" then $f(\xi)$ would become calculable. So it is this notion of indeterminate real number — a number whose construction captures the sense of indeterminacy in our intuition of time — allows Brouwer to merge modern set theory with the older viscous continuum. And it shows how Brouwer has adapted the eighteenth century notions of intuition and the continuum to his modern mathematical program.

I haven't yet addressed the third issue, Brouwer's attitude towards infinity. I will do that a bit later, and I will show that he too has a certain ambivalence here. Indeed, an inevitable ambivalence, I shall claim. But in order to make that precise, I will first point out that in fact Brouwer's definition of ξ fails, and then, using some machinery from a generalization of the Hangman paradox, I will isolate the underlying problem. This in turn will display the ambivalence in Brouwer's attitude towards infinity.

The definition fails because should any d_i in ξ's decimal expansion be chosen < 9, that would automatically entail that $\xi < 1$. But we know this from the start, and are thus constrained to choose $d_i = 9$ for all i. That in turn guarantees that $\xi = 1$. So one way or the other, ξ cannot remain indeterminate. As it happens, I've revised Brouwer's text to make this point more sharply. But I want to show now that within the intuitionistic standpoint one *cannot* define a totally indeterminate real to serve Brouwer's purposes here.

That fact will make precise his ambivalence towards infinity: his desire to have a real number that is eternally indeterminate but his inability to define one. The core of Brouwer's problem is his need to *guarantee* that ξ is indeterminate. The problem is *forced indeterminacy* which is really what the Hangman paradox is all about.

A QUICK LOOK AT THE GENERALIZED HANGMAN

The Hangman in Brief

On Sunday a judge sentences a prisoner to death by hanging with the following conditions: (i) You shall be hanged. (ii) The hanging will be at noon on either Monday, Tuesday or Wednesday. (iii) The day of your hanging will be a surprise (i.e., on the morning of that day, as far as you can deduce, you may or may not be hanged at noon.) The prisoner reasons as follows: I can't be hanged on Wednesday, for then it wouldn't be a surprise. Having excluded Wednesday, the same reasoning can now be applied to rule out Tuesday and Monday as well. So I can't be hanged.

There is an ever-growing literature on this, but I want to take a straightforward position [Compare (Gardner 1991)]. I believe that the judge's decree is inconsistent.[4] Whatever else he may have thought about himself in the past, at the moment he hears the judge's decree, the prisoner must literally count his days. But he has to count them in an unusual way. In particular, the judge's timing requirement tells him that the best he can do is map his future on the chart below ("h" represents being hanged, and "a" stands for being alive at the end of the day). The path tracking his future lies in the tree in Figure 1.

The demand for an actual hanging says that "a" cannot be at a terminal node of that path. But the surprise requirement entails that if "h" is at a terminal node on the path tracking his life, then the row in which that "h" occurs on the tree must also have an "a" as well (that expresses the claim that as far as he can deduce either event is possible).

The sequence of nodes which ends at a Tuesday hanging satisfies this last requirement. Obviously the sequence which ends with Wednesday doesn't. However if we remove that sequence, then we get a different tree; one which violates the certainty-of-hanging requirement. Indeed, there is no way to build a tree which satisfies all the judge's conditions. And that is why I believe the judge's decree to be inconsistent.

Intentional Sequences and Trees

To learn from this we need to generalize the notion of a "tracking sequence" and correspondingly generalize the notion of a tree as a set of potential such sequences. That is what I want to do now with the notions of "intentional sequences" and "intentional trees."

We can look at an intentional sequence as a sequence of ordered pairs $\{N_\alpha(i), R_\alpha(i)>\}_i$, where $R_\alpha(i)$ is the set of potential numerical values at i, and the actual numerical value $N_\alpha(i)$ is an element of that set. $N_\alpha(i)$ represents what we ordinarily simply call $\alpha(i)$.[5] It is best to view this as a generalized algorithm. In an ordinary algorithm $R_\alpha(i)$ is always a unit set.

I call these sequences "intentional sequences" because we can have a pair of sequences, α and β, which are numerically equal (i.e. for all i, $N_\alpha(i)=N_\beta(i)$) but which still don't count as identical. We shall see an example in which this actually happens in a moment.[6] But for now notice that these sequences give us a concrete handle on the notion of indeterminacy: α is *undetermined at i* if $R_\alpha(i)$ has two or more elements. And we can say α is *k-indeterminate* if it is undetermined at every i between 0 and k (inclusive of k). A fully determinate sequence is one which is determined for each natural number, i; and a fully indeterminate sequence is one which is indeterminate at each i.

An "intentional tree" is a set of intentional sequences which share some common constraint on their numerical values. The second word in the title comes from the fact that we can picture such a set by a tree-like diagram. But it is important to remember that an intentional tree is not an arrangement of points. It is a collection of intentional sequences.

Now in the Hangman story, the judge is defining the sequence α tracking the prisoner's fate. If we let 1 and 0 stand in for "a" and "h" respectively, then the timing requirement puts α in tree T_1 pictured below; the hanging requirement entails that α can never terminate with a 1, and the surprise factor, requires that if $\alpha(k)=0$, then $R_\alpha(k)=\{0,1\}$ [That is, if $\alpha(k)=0$, then α is undetermined at k].

Clearly these conditions form an inconsistent set. There is no intentional sequence α which can satisfy these requirements. The proof of this fact is just the prisoner's backwards induction.

Tree T_1 in Figure 2 is what I call a 2-high tree. There is no possible element in T_1 which remains indeterminate for more than two places. Speaking informally, a tree is *k-high* if every element of the tree either ends or is determinate by its k+1st place. Formally, an intentional tree T is *k-high* if no sequence which is an element of T is j-indeterminate for any $j < k$. As a further example, tree T_2 in Figure 3 is what I call ω-high.

Now the definitions themselves make it clear that the judge cannot require a sequence to be k-indeterminate unless he has put it into a tree which is at least k-high. Thus, for instance, there can be no 3-indeterminate element in T_1.

And from this we can give some instructions to the judge: he can incorporate an element of indeterminacy in the definition of a sequence α by placing α in a tree T which is j-high (for ($j \geq 1$), and then decreeing that there must be some $k \leq j$ such that α is k-indeterminate. In our case he can say that the hanging will be a surprise or it will be on Wednesday. That is α is k-indeterminate for some $k \leq 2$.

But if the decree also contains the requirement that α be *precisely k-indeterminate for some specific k*, then putting it into a k-high tree is not enough. This holds even if the decree prescribes that $k = \omega$. For in these circumstances, if the decree builds the exact degree of α's indeterminacy into the definition of α, then variations on the prisoner's induction will work to once again show the decree inconsistent. Using only the notion of the "height" of an intentional tree, the judge might intend to impose a k-indeterminacy, but he can't say it.

A decree can guarantee that α is k-indeterminate only if it has put α into a tree T which always branches for finite sequences of length $\leq$k. I call such a tree, k-free. Formally, an intensional tree T is *k-free* if every finite sequence in T which is of length $\leq$k has at least two distinct continuations which are in T. It allows complete freedom of choice up to the first k-choices. For example, tree T_3 in Figure 4 is a 5-free intentional tree.

A necessary and sufficient condition for the judge to be able to require that α be exactly k-indeterminate is that he puts α in a k-free tree, and then leaves things at that.[7]

We can phrase this as a general limitation on the power to define an intentional sequence. If we want the definition to contain an explicit degree, $k(>1)$, of indeterminacy, then we must put the sequence in a k-free tree. If we put the sequence only in a k-high tree, then we can at best put an upper bound on the indeterminacy.

The Continuity Theorem Again

Now let's return to Brouwer's continuity theorem. Brouwer wants to define a real number ξ, in a way that always holds open both the possibility that ξ will be equal to 1 and that ξ will be less than 1. If ξ remains thus indeterminate, then $f(\xi)$ will never be calculable, and that will do the job. But as we saw in the discussion of Brouwer and intuition, this is impossible.

The reasoning I used at the end of that discussion is actually a version of the Hangman. For Brouwer has placed ξ in an ω-high tree, but one that is not ω-free. Yet Brouwer's definition includes the demand that ξ remain fully-indeterminate, that is ω-indeterminate, and we have just seen that this is impossible![8]

This is a real problem in Brouwer's proof. Indeed, though he often echoed and developed early arguments, this particular argument never resurfaced in any of his subsequent work. He never again tried to define a totally indeterminate real number.[9] His later work uses "creative-subject" counter-examples, based on stages in the attempts to solve as yet unsolved mathematical problems (see Brouwer 1948a and 1948b). The basic idea is to define ξ so that $d_k=9$ if by the kth stage the problem is not yet solved, and $d_k<9$ if by the kth stage it is solved. When we form sequences in this way we never guarantee anything more than 1-indeterminacy. There is no promise that the sequence will ever settle down and become determinate.[10]

The Intuitionist Outlook

Now it is tempting to save Brouwer's argument by pointing out that there is a third player in the hangman story, the executioner. The judge, as we saw, specifies a tree T, to which the sequence mapping the prisoner's fate is confined; and he sets a certain degree of indeterminacy for the prisoner as the sequence unfolds. The executioner's job is to implement both aspects of that decree. But for the executioner himself nothing is indeterminate.

In particular, the executioner defines a *fully determinate* sequence, β, which is an element of the tree T and which ends at the node corresponding to the day of the prisoner's death. The prisoner, for his part, defines a sequence, α_β, which depends on β (extensionally). That is for all i, $[N_\alpha(i)=N_\beta(i)]$. If β is an element of T, this will guarantee that α_β is too. The executioner's task is to define β in such a way that the other part of the judge's decree is fulfilled as well. That is, he must see to it that α_β has the degree of indeterminacy required by the judge. The prisoner's hope is that it can't be done.

Though the executioner's sequence, β, is always determinate — he knows exactly where he is going — the prisoner's sequence, α_β, is not ordinarily determinate. The prisoner needn't know the final outcome, or even what the next value will be. Specifically, for all i, $N_\alpha(i)=N_\beta(i)$; but that does not entail $R_\alpha(i)=R_\beta(i)$. This is a perfect instance of two quite different sequences which are nonetheless extensionally equal.[11]

So we might try to save Brouwer's proof by saying that the real number 1 is given by an executioner's sequence, which is fully determinate. It is just the sequence {0.9, 0.99, 0.999, ...}. And the ξ being defined would be the prisoner's sequence, ξ_β, which is dependent on this executioner's sequence but whose definition is "unaware" of β's ultimate destination [see (Troelstra 1977)]. It is tempting, as I said, to interpret Brouwer's argument that way, but it would be wrong.

In the intuitionistic scheme of things, there is no "behind the scenes," and no role for an "executioner." The central thesis of mature intuitionism is that when we are dealing with mathematical objects there is no executioner's perspective. We are not playing games here, withholding information from ourselves to maintain an artificial indeterminacy. Brouwer's notion of a choice sequence is designed to show what happens when we do mathematics from within the prisoner's perspective. Brouwer's examples using these sequences all invoke an element of contingency which forces us to acknowledge the contingent facts that develop as a sequence α progresses. We cannot escape this, and we cannot escape the resulting indeterminacy. There is no room here for an additional extensional player for whom everything is determined.

Indeed, "classical" mathematics is just mathematics done from the executioner's perspective, according to which the actual temporal progression through the stages of his sequence, β, has no bearing on the truth of propositions like $\beta=1$ or $\beta<1$. These propositions and everything else that is relevant about β are all determined from the start.

We can say that the executioner has a classical theory of truth about his objects, while the prisoner's perspective is the backbone of an anti-realist position, according to which truths come into being as discovered. The latter is clearest in the Beth and Kripke models that provide a formal semantics for intuitionistic logic. The simple Kripke-model for instance that shows us the difference between $\forall x\exists yB(x,y)$ and $\forall x\sim\sim\exists yB(x,y)$ is schematically pictured in Figure 5. This is really just an ω-free tree, where the branches above each node represent the different ways in which the existential quantifier can be witnessed. From the executioner's perspective, of course, these witnesses are known from the start, and there is no ground to refrain from asserting $\forall x\exists yB(x,y)$.

Brouwer on Infinity

This way of putting the philosophy of intuitionism allows us finally to make Brouwer's attitude toward infinity quite precise, and in so doing it shows how we can use Brouwerian intuitionism to interpret Kant's own attitudes about infinity.

1. Regarding Brouwer's notion of infinity: notice that on the one hand if we were

dealing with propositions that we knew to be finitely decidable (about objects, say, which always go into finitely high trees) then we would treat those propositions as already decided. In that circumstance, there would in fact be no appreciable difference between the prisoner's and executioner's perspectives.[12] So the very distinction that characterizes intuitionism — the distinction between the prisoner's and executioner's perspectives — presupposes the existence of ω-indeterminate sequences. On the other hand, we have seen (and Brouwer himself came to recognize) that these are sequences whose existence cannot be proved from the prisoner's perspective. Intuitionists suffer a tension here: they can't help presupposing the possibility of the ω-indeterminate sequences whose existence they cannot prove from their prisoner's point of view. This is Brouwer's ambivalence towards infinity.[13]

2. This, indeed, is how Brouwer can help us with Kant, for this intuitionistic tension is a precise version of a tension that Kant (in the "Antinomy" chapter) says is inevitable.

Think of the "First Antinomy," the one about spatial and temporal magnitude of the world. The point of the antinomy is that though we must think about the world-whole (with an infinite spatial and temporal magnitude), nevertheless we don't have the epistemic resources to prove its existence.[14] Indeed the intuitionistic negation in $\forall x{\sim}{\sim}\exists y B(x,y)$ gives a precise sense to the idea that the infinite size of the world is a regulative principle. It gives sense to the view that for each part of the world that we find we are enjoined to search for yet a further part, and that we are assured success yet cannot assert the existence of the further part until we actually find it. The extent of the world (as produced in these searches) is an ω-indeterminate sequence in the Kripke structure which verifies this formula!

KANT AND BROUWER DISTINGUISHED

A Closer Look at Kant on Intuition and Infinity

But before we hoot and crow about the neat way that Brouwer has melded the old with the new in mathematics, and has made precise Kant's oft maligned thinking in the "Antinomy," we had better attend to the deep differences between Kant and Brouwer.

For one thing, Kant doesn't assume that this world-whole, given by an ω-indeterminate sequence, actually exists. In fact, he denies that it is even an object at all. It is, in his eyes, merely an aggregate, given by a "general concept" and not at all by a singular intuition (A519/B547). Moreover, this does nothing to soften the conflict with the "Aesthetic," where he says that space is given by an intuition and is indeed an infinite magnitude. The answer comes from looking more closely at Kant's notion of intuition. Speaking phenomenologically, for Kant, an empirical intuition is a core of sensory content organized by a sortal concept. But speaking epistemologically, it is simply an adequate grasp of an individual object.

Kant's view of intuition is heir in an important sense to Leibniz's notion of God's grasp of the complete concept of an individual substance, a representation from which all truths about the object are ultimately derived. The Leibnizian mark of a true individual (as opposed to a mere aggregate) is the completeness of its concept with regard to all elementary predicates:

> It is the nature of an individual substance or complete being to have a concept so complete that it is sufficient to make us understand and deduce from it all the predicates of the subject to which the concept is attributed.(Leibniz 1686, VII).

Only something that is thus completely determined is truly unified and truly counts as an object. And Kant inherits this as well: a true object must be fully determined in all elementary properties.

> But every *thing*, as regards its possibility, is likewise subject to the principle of *complete* determination, according to which if *all the possible* predicates of things be taken together with their contradictory opposites, then one of each pair of contradictory opposites must belong to it. (A571-2/B599-600)

And a legitimate intuition must provide the grounds for knowledge of all those properties.

Now to be sure, Kant differs from Leibniz in one important respect: Kant does not require an intuition to itself provide all the information that there is about its object. Kant's point is rather that the presence of the intuition *guarantees* that the information can be gotten. For Kant, intuitive knowledge itself is an attenuated, sequential business.[15] But still, Kant insists that the attenuation is always finite. Once an object is intuitively given, then there will be no question about that object which must remain forever unanswered. Without that guarantee, we have only a general concept, and not an intuition.

This explains Kant's claims about the first "Antinomy." The scientist's grasp of the "world-whole" is not intuitive.

> In natural science, ... there is endless conjecture, and certainty is not to be counted upon. For the natural appearances are objects which are given to us independently of our concepts, and the key to them lies not in us and our pure thinking, but outside us; and therefore in many cases, since the key is not to be found, an assured solution is not to be expected. (A480-1/B508-9)

We don't after all know yet what further objects there are. We must wait until we find them. So the "world-whole" is not an object, with firm size, but merely an aggregate, without a size.

The same would go for any alleged mathematical object whose properties are not all determined. It wouldn't be a legitimate object. Indeed, when we turn to mathematics, we find Kant asserting quite clearly that there is no indeterminacy or uncertainty here:

> It is not so extraordinary as at first seems the case, that a science should be in a position to demand and expect none but assured answers to all the questions within its domain,... although up to the present they have perhaps not been found. In addition to transcendental philosophy there are two pure rational sciences, ... namely, pure mathematics and pure ethics. (A480/B508)

This holds because the mathematician abstracts from the receptive aspect of empirical objects. For that reason, his grasp of mathematical space *is* epistemically complete and therefore is intuitive. Mathematical space can be grasped as an object. And that is his point in the "Aesthetic." Unlike the empirical world, space does have a magnitude, and an infinite one at that.

It is not that we somehow "picture" infinite mathematical space. It is rather that we have an intuitive grasp of the infinite process for measuring and describing further and further regions. And since there is no more to mathematical space than what is given by this process — this is Kant's "transcendental idealism" — that grasp must suffice for a complete determination of all the mathematical properties of space.

This is the executioner's perspective, *par excellence*. Once a mathematical object is defined, be it finite or infinite (space itself, for instance, as we have seen), then all its properties and all of its relations to other relevant objects are fixed and determined from the outset. Moreover, the executioner is not hiding behind the scenes here. For Kant, the mathematician is the executioner as well as the judge. To be sure, our vision at the moment is often limited (like the prisoner's). There always are some unanswered questions. But, says Kant, there can be no eternal ignorance, and thus the limitations of the moment have no ultimate epistemic interest.

I must also say that we find in Kant none of the Brouwerian ambivalence towards infinity. He has, in fact, quite a straightforward attitude here: infinity has no place in empirical science, and it is perfectly fine (and fully graspable) in mathematics. Moreover there is none of the intuitionistic internal tension towards infinitely indeterminate objects. For Kant there simply are no such mathematical objects. That too is part of the executioner's perspective.

So Kant on this crucial aspect of intuition — the question of epistemic completeness — does indeed have quite a different view than Brouwer. That is one of the points that I promised at the outset. And this difference does, as I also suggested, lead Kant to the classical, executioner's perspective rather than some limited prisoner's point of view with respect to mathematical infinity.

A Closer Look at Kant on the Continuum

It might seem that Kant and Brouwer do agree here, for both advocate the thick, viscous continuum. But despite this similarity, here too they will differ. For Brouwer, as we have seen, constructs the continuum set theoretically, while Kant imposes viscosity by rejecting the idea of a continuum built up out of independently given elements, points. Indeed, you may ask, if I am right that Kant does advocate the executioner's perspective in mathematics, why then does he insist on the Aristotelian viscosity of space, something that in Brouwer comes from the prisoner's perspective? Why indeed does he reject the notion of pre-existing, separable points? It couldn't be because we can't picture such small things. Infinite space in the large is no more picturable. Nor is it a problem with the infinite processes that we need in order to produce points from sequences of nested intervals or convergent sequences of rational numbers. For we need to be able to grasp an infinite process in order to determine the magnitude of space. And, once again, there is nothing intrinsically more graspable about one such process than there is about the others.

The problem for Kant is that we need to be able to conceptually describe these processes. We have to be able explain in general terms how we go from an originally given object to the mathematical "point" in question. Or to put it as I did above, we need a "sortal concept" in order to individuate the process. But Kant believed that there aren't enough explicit mathematical definitions to capture all the possible processes. To put it aphoristically, in a continuum more points can be described than can be defined.

Actually Kant is following Euler here: I am thinking about Euler's view on the vibrating string problem, and his claim that we do not have enough explicitly definable functions to express all possible initial states of the string. Points given by values of functions exceed our algebraic expressive power.

> The various similar parts of the curve are therefore not connected with each other by any law of continuity, and it is only by the description that they are joined together. For this reason it is impossible that all of this curve should be included in any equation. (Euler, 1755).

Commentators emphasize that the "discontinuous" functions that Euler wants to add are really continuous in the modern sense. (His discontinuous functions are those given by more than one analytic expression.) But the heart of his position is that there is no explicit concept of function which covers all cases in which we can individuate a point [This is part of his debate with Daniel Bernoulli; see (Yushkevich 1976) and (Bottazzini 1986)]. And Kant follows him on this. So for Kant the only tool we have which will do that job is the concept of free motion.

And this, in the end, is a deep difference between Kant and Brouwer. It is an epistemological difference: Brouwer says that his definition of an indeterminate real number is a legitimately intuitive presentation of a number. Kant says it is not an intuition at all. (And it doesn't matter whether it is ω-indeterminate like ξ attempted to be, or only 1-indeterminate.) It is an ontological difference as well. For Kant, the description of such an indeterminate entity represents an aggregate and not an object at all [see (Kant 1992, I, 1, 15n)].

So even on the issue of the thick continuum — the area in which Brouwer most closely mimics the eighteenth century — we find, when we probe beneath the surface, more difference than similarity.

CONCLUDING SPECULATIONS

I don't want to adjudicate this dispute, but rather let me mention some speculative conclusions that are suggested by this case study. In particular I would like to take a lesson from our experience with Kant and Brouwer that may elucidate the grounds for the classical approach to the continuum. I will also try to show how our case study may tell us some things about mathematical progress and may help us refine Prof. Cellucci's notion of an "open world view."

The Equilibrium Hypothesis

The pressing issue concerning the continuum is the question of how we can distinguish the notion of continuity from mere infinite divisibility. Kant, we have seen, claimed that viscosity, or inseparability, is the distinguishing mark of continuous manifolds. And he imposed this viscosity by postulating a special independent intuition of "flowing." Brouwer at first embraced this view that there is a special sense of continuity, but then abandoned it when he found a set theoretic way to recover viscosity. He did this, as we have seen, by assuming the prisoner's perspective not only epistemologically but ontologically as well. He introduced "incomplete" objects, real numbers which may simply lack precise location and other relevant properties.

Now suppose for the moment that we agree with Brouwer that some of the mathematical objects which we define may indeed be incomplete. And suppose we also agree to conduct our mathematics on the assumption that mathematical domains include such objects. We have seen that this assumption itself presupposes that there may be fully ω-indeterminate elements in these domains. The Hangman paradox then tells us that we can never prove that such an object exists. And so the lesson I want to take is that doing mathematics from Brouwer's "prisoner's perspective" rests on this "disequilibrium" between our ontological assumptions about what there is and our epistemological limits to what we can prove to exist. Kant's picture of mathematics, I have suggested, embraces the opposite view. For Kant, what there is corresponds exactly to what we can constructively grasp. This equilibrium is the heart of Kant's transcendental idealism. It also suggests a more general Kantian conjecture about the continuum.

The conjecture is that Brouwer's disequilibrium is really at the core of his thick continuum, and thus also that the thin "classical" continuum is characterized more by the Kantian equilibrium between ontological reach and epistemic grasp rather than by necessarily infinitary or platonistic notions. The idea, then, is that given some characterization of the limits of our mathematical abilities, we can still have a classical (thin) continuum (and discontinuous functions) so long as we are careful to assume that the only existing objects are those which can be constructed within those limits.

The conjecture is quite general. We do not have to limit ourselves to some particular epistemic conception, Kant's or Brouwer's or anyone else's. The point is that the classical continuum stems from the equilibrium between epistemology and ontology and not from some broad or narrow part of the epistemology itself. That equilibrium is, if you like, the general framework bequeathed to us from Kant's transcendental idealism.

Much needs to be done in order to establish this conjecture rigorously, but for now let me simply cite two pieces of supporting evidence.

First, recursive analysis. Suppose that we admit only recursive functionals as real valued functions; but suppose also that we allow these functions to range over the full set of real numbers (both the recursive and non-recursive ones). The explicitly defined functions are methods for constructing real numbers. Indeed, the recursive reals are those thus constructible when we start with the rationals. We are now assuming however that there are real numbers in the universe of discourse which cannot be constructed. This is a clear parallel to Brouwer's disequilibrium. And as it turns out, we can prove under these assumptions that all the total functions are continuous. (For, just as in Brouwer's case, they will not be able to discriminate among very close real numbers in their domains.) That is the equivalent of a thick continuum. When, on the other hand, we restrict the domain to recursive reals, then discontinuous functions — and a thin continuum — reappear [See (Rogers 1967, 315)].

The second bit of evidence is just V=L: This is Gödel's set theoretic assumption that the universe of existing sets is identical to the universe of constructible sets. It lies at the heart of his proof in his (1940a) of the consistency of the continuum hypothesis and of the axiom of choice with the axioms of set theory, and it is, of course, an exact parallel to our assumption of equilibrium. Indeed, in reporting his motivation for this axiom in (Gödel 1940b) Gödel says that the driving consideration was his belief that the well ordering theorem (the paradigm of a thin continuum) would follow if we could assure that all real numbers were definable. This simply supposes that there is a balance between what is assumed to exist and that which can be epistemically grasped.

As I said, the equilibrium hypothesis awaits a rigorous proof. But these partial indications do lend support to the view that once we remove Kant's independent allegiance to the thick continuum, then his deeper philosophical position actually provides a basis for the classical rather than the intuitionistic theory of the real line.

Mathematical Progress and "World Views"

Progress

I began this essay with a look at Russell's assessment of mathematical progress. It was for him — and indeed for many of us — a linear progression from Kant's time to our own. Kant had intuition, a thick continuum, and an inadequate (indeed, inconsistent) account of infinity. We supposedly have gone beyond all of that. We have abstract mathematics freed from intuition, a new acceptance of infinity, and a set-theoretic and thus thin continuum. Our improvements, he believed, result from the newfound power of symbolic languages. Infinity, for instance (both the infinitely large and the infinitely small), can be expressed by the $\forall\exists$ quantifier combination. The improvement is patently clear. And so, Russell believed that if Kant had possessed this power of expression he would not have been driven to base mathematics upon distilled human intuition, nor to embrace the viscous "flowing" image as the main characterization of the continuum.

But the present case study teaches us that Russell's picture is wrong. It is wrong on each of the counts, and Brouwer is the proof. Brouwer proved that the set theoretic approach to mathematics does not automatically banish an appeal to intuition; nor does it mandate a well orderable, "thin" continuum. For he managed to preserve the thick, viscous continuum — a continuum which rests upon the *a priori* intuition of time — within the set-theoretic context. And he did this in a way that can well be described using Russell's own symbolic representation of infinity, formulas employing the $\forall\exists$ quantifier combination.

Indeed, the true dispute between Russell and those who favor an "intuitive" foundation for mathematics does not revolve around our ability to express infinity in formal languages. The dispute rests, rather, on how that $\forall\exists$ quantifier combination is to be interpreted. Russell sees a formula like $\forall x\exists yB(x,y)$ as expressing the existence of a set of arbitrarily combined pairs; by contrast the intuitionist views this formula as claiming the existence of a process — indeed, a process which is tracked as a path in what I have called an intentional tree.

And the fact is that rather than undermining Kant's view of infinity and intuition, this modern notation serves instead to explicate Kant's view. For as we have seen the modern intuitionistic modeling of this sort of formula expresses Kant's views about empirical measurement and the size of the empirical world with a precision that Kant could not attain. And the precise difference between $\forall x \exists y B(x,y)$ and $\forall x {\sim}{\sim}\exists y B(x,y)$ corresponds exactly to Kant's distinction between the infinitude of mathematical space and the indefinite extent of physically occupied space.

What we can say, then, is that our modern techniques have given us tools to express more clearly, and thus better to understand, the ideas which lie at the heart of this early view of mathematics. Our modern perspective allows us to state more precisely than Kant and his contemporaries could have done what their underlying assumptions are, what consequences those assumptions have, and what choices must be made in order to preserve the different philosophical and mathematical priorities. Much the same may hold of my explication of Brouwer's intuitionism. The machinery that I sketched in Part II above — machinery that comes from a surprising area of philosophical logic — helps us state Brouwer's point of view with increased precision and may explain the factors causing him to revise his theory of choice sequences. But these explications alone do not dictate mathematical or philosophical changes. Kant may well have expressed himself differently had he had access to our new more precise mathematical tools, but Russell is wrong in thinking that this alone would have changed Kant's views about the intuitive basis of mathematics or about the viscosity of the continuum. Those deep changes must be won on other grounds, including philosophical grounds about the nature of intuition and the very concept of an object. And if the "Equilibrium Hypothesis" is correct then we can see more clearly the available strategies which allow one to keep the classical continuum and still maintain allegiance to intuition or "constructivity".

So though Russell was wrong about the simple linear progression here, we certainly do have a methodological lesson here. Using methods that were either unrelated or unavailable we can see the picture more precisely than the original players. And we can use that clearer perspective — with an eye, so to speak, towards the past as well as the present — in order to map out our current mathematical and foundational programs. All of this is a strong sort of mathematical progress.

"Open" and "Closed" World Views

This last suggestion — proposing, as it does, an interplay between new concepts and methods on the one hand and backwards glances on the other — is reminiscent of the "analytic method" that Professor Cellucci (in the present volume) associates with the "open world view." And I want to end with a few words about that.

Professor Cellucci contrasts what he calls closed and open mathematical "world views." Adherents of the closed world view emphasize the systematic unity of mathematics — ultimately the unity of a formal system. They see mathematical truth as a set of consequences of explicit axioms. The open world view, by contrast, emphasizes conceptual change in mathematics, and thus a certain unpredictability. So its advocates have a tolerance for incomplete pictures and evolving mathematical methods. Professor Cellucci takes Gödel's Incompleteness Theorem as the death knell of the closed world view, and he chides Gödel for what appears to be an inconsistent optimism that all mathematics is still formalizable. Indeed I should add, though Professor Cellucci does not, that Gödel in fact seemed committed to the even more surprising optimism which claims that there are no ultimately undecidable mathematical statements [see (Gödel 1951)].

Now at first sight Kant and Brouwer appear to be perfect exemplars of these opposing visions, Brouwer embodying the open world view and Kant the closed. Indeed, we may say that Brouwer — perhaps more than anyone else — builds the open world view into his explicit mathematics. His choice sequences, after all, are designed to capture the sense of an unpredictable open future. And, we are tempted to say, the prisoner's perspective that characterizes intuitionism is a direct philosophical consequence of this openness towards the future.

As for Kant, Professor Cellucci himself credits Kant as the godfather of "closed" systematicity, and provides us with ample citations to underscore this accusation. These are passages in which Kant talks about systematicity, about mathematical unity and about how mathematical consequences are contained from the start the way parts are contained in an as-yet-undeveloped "germ." [A834/B862] And Kant's executioner's perspective lends further credence to this assessment of his closed world view.

Yet I want to dispute this picture of Kant, and I want to use my grounds for this dissent to suggest a refinement of Professor Cellucci's distinction and, indeed, an explanation of Gödel's position.

First the dissent. Look back for a minute at Kant's reason for assuming the executioner's perspective in mathematics. At its heart there was a certain optimism about our ability to solve all mathematical problems. This in turn led him to grant a certain intuitive unity to mathematical domains, the same intuition-based unity that characterizes our grasp of ordinary objects — the epistemic closure that allows this grasp to count as presenting an object. But Kant would never claim that this unity is somehow a deductive closure. That is Leibniz's view, not Kant's! For Kant our intuitive grasp of an object promises that the answers to our questions about that object can be found. But Kant does not claim that those answers are deductive consequences of either the matter or the form of the intuition. To find those answers we must await further sensory input, and sometimes we must even await the evolution of adequate scientific concepts.[16]

What we need here, in fact, is Kant's distinction between intuitions and concepts. Intuitive grasp for Kant does give an *epistemic closure*, the promise of answers to all of our questions about the grasped object or domain. This is the closure in which the "unity of the whole" does precede the grasp of its parts. Indeed, we may have an intuitive grasp of the whole even before we have the conceptual means for describing the component parts.[17] Conceptual grasp, by contrast, provides what we may call *combinatory closure*, the closure of a formal system. A well-defined concept gives us the marks by which to distinguish a class of objects, and the certainty that any deductive consequence of those marks must characterize each object in the class. Anything not covered by those marks is undetermined by the concept.

For Kant, this distinction carries over *mutatis mutandis* to mathematics. To say that a branch of mathematics is conceptually (combinatorially) closed is to say that we have a full of set of basic concepts for describing the objects of that branch of mathematics and their properties. To say this would indeed amount to claiming that all the truths are merely deductive consequences of a small set (or perhaps a recursive set) of basic axioms. But one lesson that we must take from Kant's Eulerian treatment of the continuum is that we do not have such a set of basic mathematical concepts! There is more to space than can be described by any given set of explicit concepts.

This then is the refinement that I would like to suggest for Professor Cellucci's distinction. We need to distinguish between combinatorial closure on the one hand — the closure, as I said, of a formal axiomatic system — and the optimism of epistemic closure on the other — the sense that all mathematical problems are subject to solution (something that Brouwer would deny). This epistemic closure is actually quite hospitable to the sort of conceptual development and interfield cooperation that I have been explicating in the body of this essay.

It is only combinatory closure that Gödel's theorem undermines. That theorem would undermine our faith in epistemic closure only if we also believed that some formal system or other contained all the basic concepts for expressing mathematics. Gödel himself never claimed that. He advocates formalization only after the fact, so to speak. For he believes that there are no mathematical facts that cannot be described and defended. This too is a Kantian legacy. We have seen that the heart of Kant's idealism is just this identification of what is true with what can (eventually) be described and known.

NOTES

1. See (Russell 1897 and 1903). An updated version of the Russellian critique (though one which attempts to be sympathetic to Kant) is (Friedman 1992).
2. Actually he gave a non-set-theoretic presentation of the continuum, which is quite close to Kant's: "Having recognized that the intuition of "fluidity" is as primitive as that of several things conceived as forming a unit together, the latter being at the basis of every mathematical construction, we are able to state properties of the continuum as a "matrix of points to be thought of as a whole". ... However the *continuum as a whole* was given to us by intuition; a construction for it, an action that would create from the mathematical intuition "all" its points as individuals, is inconceivable and impossible." (Brouwer, 1907). But within ten years he had completely adopted the set theoretic construction of the continuum.
3. Thus in (1912) he says: "But the most serious blow for the Kantian theory was the discovery of non-euclidean geometry, a consistent theory developed from a set of axioms differing from that of elementary geometry only in this respect that the parallel axiom was replaced by its negative. For this showed that the phenomena usually described in the language of elementary geometry may be described with equal exactness, though frequently less compactly by in the language of non-euclidean geometry; hence it is not only impossible to hold that the space of our experience has the properties of elementary geometry, but it has no significance to ask for *the* geometry which would be true for the space of our experience. .... However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant's *a priority* of space but adhering the more resolutely to the *a priority* of time."

4. This is close to the position advocated by (Kaplan and Montague 1960). But one of the advantages of the present approach is that we needn't invoke the machinery of Tarski's theorem that they use.
5. An *Intentional Sequence*, α, is a sequence of ordered pairs $\{N_\alpha(i),R_\alpha(i)\}_i$, where $N_\alpha(i)\in R_\alpha(i)$. $D(\alpha)$ is the set of propositions which determines the sequence $\{R_\alpha(i)\}_i$.
6. I also want to deflect any indication of how the elements $N_\alpha(i)$ are generated. In particular the notion is independent of any commitment to "free choice" or to any other process for generating the numerical values.
7. Formally, this is the content of Propositions 1, 2 and 3: *Proposition 1*: If $D(\alpha)$ places α in a tree, T, then α can be k-indeterminate (for $k\geq 1$) only if T is j-high for some $j\geq k$. *Proposition 2*: If T is k-free, then we can define an α in T such that $D(\alpha)$ entails that α is k-indeterminate. *Proposition 3*: If $D(\alpha)$ entails that α is an element of a tree, T, and that α is k-indeterminate, then T is j-free for some $j\geq k$. Now every k-free tree is k-high. But all we can get conversely is that every k-high tree is at least 1-free. This does show, however, why the judge can set 1 as a lower bound for indeterminacy.
8. Technically, it contravenes Proposition 3.
9. Indeed, in a footnote to (1952) he appears to acknowledge this problem explicitly. He says: "In former publications I have sometimes admitted restriction of freedom with regard also to future restrictions of freedom. However this admission is not justified by close introspection, and, moreover, would endanger the simplicity and rigour of further developments."
10. Hand in hand with that is a strict definition of calculability for f: $f(\alpha)$ must be calculable on the basis solely of the definitional information $D(\alpha)$.
11. Indeed, the only method to state existence of a k-indeterminate sequence in a tree which is not k-free is to use a (fully determinate) executioner's sequence. That is the point of Proposition 4. *Proposition 4*: If T is a k-high tree, then for each n ($1\geq n\geq k$) there is an executioner's sequence β ($\beta\in T$) such that the associated prisoner's sequence, α_β , is precisely n-indeterminate.
12. To be precise: In order to distinguish the prisoner's perspective from the executioner's perspective in this finite case, we would have to restrict the prisoner to a narrow "here-and-now-ism" which prohibits him from considering any information about the future which is not already in hand. Michael Dummett makes a point quite like this in his (1977, Ch. 7) and (1978a).
13. The Hangman Paradox does not depend upon infinity in any strong way. It rests simply on the impossibility of forcing a k-indeterminacy in a tree that is not k-free. But for Brouwer, it is only the possibility that $k=\omega$ which gives bite to the problem.
14. Quite similarly, in the "Second Antinomy," we can't help postulate the infinite divisibility of matter. But again, epistemically, we cannot assert that.
15. Indeed, this replacement of the Leibnizian notion that God fully grasps a complete concept with the Kantian principle that our intuitive knowledge proceeds over time in a piecemeal fashion -- and the consequent replacement of actually complete knowledge with merely the guarantee of completion -- this is the heart of Kant's "Copernican Revolution." See my (1991) and (1995) for a further discussion of this point.
16. This last point is the lesson of the "Second Antinomy." Kant says there that when we divide a spatially extended object into ever small parts, we always have sufficient grasp of the "matter" of those parts; since that is already given in our intuition of the initial object. (For this reason, he tells us at A523-4/B552-3, the regress to ever smaller parts is a regress *in infinitum*, and not merely a regress *in indefinitum*, as in the "First Antinomy.") But we nevertheless cannot claim the existence of the smaller parts until we have developed concepts which are sufficient to describe these smaller parts.
17. Indeed, in the passage at A834/B862 where Kant uses the image of an undeveloped germ, he is actually describing a situation in which we have an inkling of a scientific field without yet having the means to fully articulate that idea.

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FIGURES

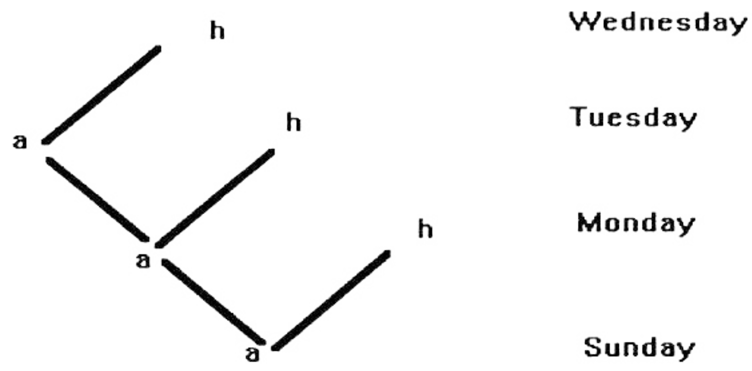


Figure 1

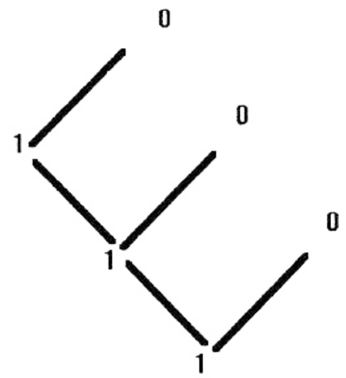
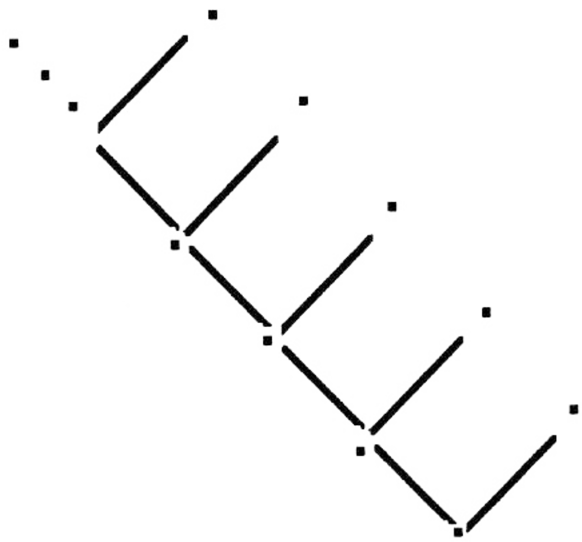


Figure 2: Tree T_1



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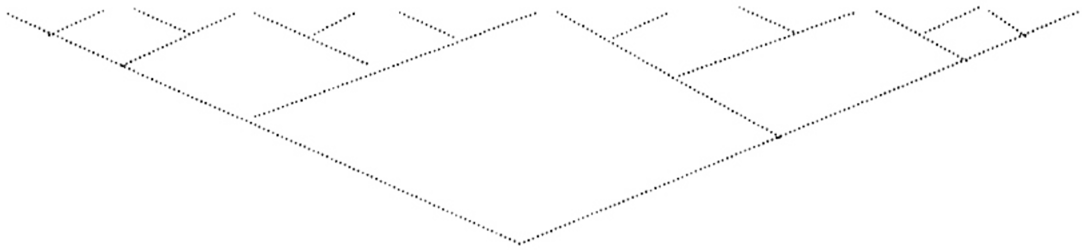

Figure 4: Tree T_3

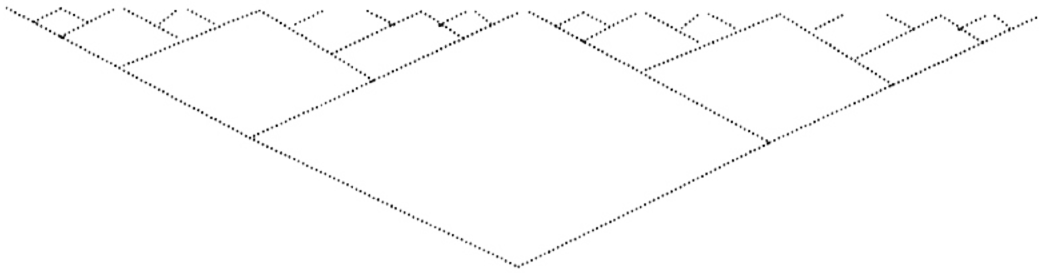

Figure 5