Fibration of context and fibration of toposes

YaMCATS, Sheffield

Sina Hazratpour
sinhp@github.io

21 March 2018

Outline of the talk

	Fibrations of	Fibrations in
Categories	Grothendieck fibrations (SGA 1) Street (aka weak) fibrations	
Strict 2-categories	Hermida	R.Street
Bicategories	Bakovic Buckley	R.Street P.Johnstone

Outline of the talk

- Ross Street (1974). "Fibrations and Yoneda's lemma in a 2-category". In: *Lecture Notes in Math., Springer, Berlin* Vol.420, pp. 104–133
- Claudio Hermida (1999). "Some properties of Fib as a fibred 2-category". In: vol. 134, pp. 83–109
- Igor Bakovic (2012). "Fibrations in tricategories". In: *93rd Peripatetic Seminar on Sheaves and Logic, University of Cambridge*
- Mitchell Buckley (2014). "Fibred 2-categories and bicategories". In: vol. 218, pp. 1034–1074
- Peter Johnstone (1993). "Fibrations and partial products in a 2-category". In: *Applied Categorical Structures* vol.1

Outline of the talk

- In the first section I will very briefly recall essentials of theory of Grothendieck fibrations.

Outline of the talk

- In the first section I will very briefly recall essentials of theory of Grothendieck fibrations.
- In the second section I will mention how to generalize from Grothendieck fibration to 2-fibration of 2-categories and bicategories.

Outline of the talk

- In the first section I will very briefly recall essentials of theory of Grothendieck fibrations.
- In the second section I will mention how to generalize from Grothendieck fibration to 2-fibration of 2-categories and bicategories.
- The third section will be about internalization of Grothendieck and Street fibrations in 2-categories. Here, I will talk about some recent work Steve and I did on reformulating internal fibrations by cartesian cells of a certain 2-functor.

Outline of the talk

- In the first section I will very briefly recall essentials of theory of Grothendieck fibrations.
- In the second section I will mention how to generalize from Grothendieck fibration to 2-fibration of 2-categories and bicategories.
- The third section will be about internalization of Grothendieck and Street fibrations in 2-categories. Here, I will talk about some recent work Steve and I did on reformulating internal fibrations by cartesian cells of a certain 2-functor.
- In the last section, I will tell how this reformulation helped us proving a result about (op)fibration of toposes.

A notation guide

I will stick to these more or less!

- For categories: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ..., $\mathcal{X}$
- Objects of categories, 2-categories, bicategories: A, B, ..., Z
- For functors: $\mathfrak{p}$, $\mathfrak{q}$, ...
- For bicategories: $\mathbb{C}$, $\mathbb{K}$, $\mathbb{L}$, ..., $\mathbb{X}$
- For 2-functors and bifunctors: $\mathbb{P}$, $\mathbb{C}\mathbf{od}$, ...
- Comma category: $\mathcal{C}^{\downarrow}$

Q: What is a fibration of toposes?

A: It is a fibration in the 2-category of toposes.

Q: What is a fibration of toposes?
A: It is a fibration in the 2-category of toposes.
We have to clarify

- what we mean by the 2-category of toposes.
- what we mean by fibration internal to a 2-category.

The 2-category of Grothendieck toposes

- The 2-category $\mathfrak{GTop}$ of Grothendieck toposes is specified by the following data:
- 0-cells are of the form

$$\begin{array}{c} \mathcal{E} \\ {\scriptstyle p}\big\downarrow \\ \mathcal{S} \end{array}$$

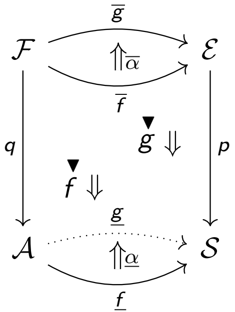
$\mathcal{E}$, $\mathcal{S}$: elementary toposes, and p : bounded geometric morphism.

- 1-cells from q to p are of the form $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$, where

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\overline{f}} & \mathcal{E} \\ {\scriptstyle q}\big\downarrow & \overset{\blacktriangledown}{f} \Downarrow & \big\downarrow{\scriptstyle p} \\ \mathcal{A} & \xrightarrow[\underline{f}]{} & \mathcal{S} \end{array}$$

$\overset{\blacktriangledown}{f} : p\overline{f} \Rightarrow \underline{f}q$: isomorphism geometric transformation.

- 2-cells between any two 1-cells f and g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} : \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$ are geometric transformations



in such a way that the obvious diagram of 2-cells commutes.

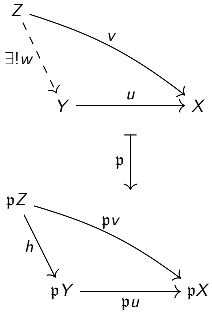
Cartesian morphisms

Suppose $\mathcal{X}$ and $\mathcal{C}$ are categories and $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a functor. A $u \colon X \to Y$ in $\mathcal{X}$ is called **$\mathfrak{p}$-cartesian** whenever the following commuting square is a pullback diagram in **Set** for each object Z in $\mathcal{X}$:

$$\begin{array}{ccc} \mathcal{X}(Z,X) & \xrightarrow{u\circ -} & \mathcal{X}(Z,Y) \\ {\scriptstyle \mathfrak{p}_{Z,X}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathfrak{p}_{Z,Y}} \\ \mathcal{C}(\mathfrak{p}Z,\mathfrak{p}X) & \xrightarrow[\mathfrak{p}(u)\circ -]{} & \mathcal{C}(\mathfrak{p}Z,\mathfrak{p}Y) \end{array}$$

More explicitly, $u : X \to Y$ in $\mathcal{X}$ is cartesian iff for any $\mathcal{X}$-morphism $v : Z \to X$ and any $h : \mathfrak{p}(Z) \to \mathfrak{p}(X)$ with $\mathfrak{p}(u) \circ h = \mathfrak{p}(v)$, there exists a *unique* lift w of h such that $u \circ w = v$.

More explicitly, $u : X \to Y$ in $\mathcal{X}$ is cartesian iff for any $\mathcal{X}$-morphism $v : Z \to X$ and any $h : \mathfrak{p}(Z) \to \mathfrak{p}(X)$ with $\mathfrak{p}(u) \circ h = \mathfrak{p}(v)$, there exists a *unique* lift w of h such that $u \circ w = v$.



Basic well-known facts about cartesian morphisms

- Cartesian lifts, if they exists, are unique up to unique isomorphism.

Basic well-known facts about cartesian morphisms

- Cartesian lifts, if they exists, are unique up to unique isomorphism.
- Any isomorphism is cartesian.

Basic well-known facts about cartesian morphisms

- Cartesian lifts, if they exists, are unique up to unique isomorphism.
- Any isomorphism is cartesian.
- Any cartesian vertical arrow in $\mathcal{X}$ is an isomorphism.

Basic well-known facts about cartesian morphisms

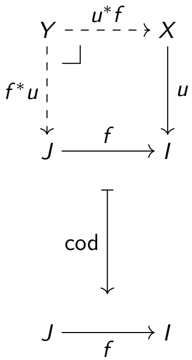
- Cartesian lifts, if they exists, are unique up to unique isomorphism.
- Any isomorphism is cartesian.
- Any cartesian vertical arrow in $\mathcal{X}$ is an isomorphism.
- Let $v\colon Y \to W$ be a $\mathfrak{p}$-cartesian morphism in $\mathcal{X}$. Any morphism $u\colon X \to Y$ is $\mathfrak{p}$-cartesian if and only if $v \circ u\colon X \to W$ is $\mathfrak{p}$-cartesian.

Typical example

- For any category $\mathcal{C}$, there is a codomain functor $\mathrm{cod} : \mathcal{C}^{\downarrow} \to \mathcal{C}$ which sends an object $f : J \to I$ of $\mathcal{C}^{\downarrow}$ to its codomain C and sends a morphism $\langle v, u \rangle : g \to f$ of $\mathcal{C}^{\downarrow}$, i.e. a commuting square, to f.

Typical example

- For any category $\mathcal{C}$, there is a codomain functor $\mathrm{cod} : \mathcal{C}^{\downarrow} \to \mathcal{C}$ which sends an object $f : J \to I$ of $\mathcal{C}^{\downarrow}$ to its codomain C and sends a morphism $\langle v, u\rangle : g \to f$ of $\mathcal{C}^{\downarrow}$, i.e. a commuting square, to f.
- Cartesian morphisms in $\mathcal{C}^{\downarrow}$ are precisely pullback squares in $\mathcal{C}$.



Fibrations of categories

Definition

A functor $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a **Grothendieck fibration** whenever for each $X \in \mathcal{X}$, every morphism $A \xrightarrow{f} \mathfrak{p}X$ in $\mathcal{C}$ has a cartesian lift in $\mathcal{X}$.

Fibrations of categories

Definition

A functor $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a **Grothendieck fibration** whenever for each $X \in \mathcal{X}$, every morphism $A \xrightarrow{f} \mathfrak{p}X$ in $\mathcal{C}$ has a cartesian lift in $\mathcal{X}$.

Example

$\mathrm{cod} : \mathcal{C}^{\downarrow} \to \mathcal{C}$ is a fibration iff $\mathcal{C}$ has all pullbacks.

Fibrations of categories

Definition

A functor $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a **Grothendieck fibration** whenever for each $X \in \mathcal{X}$, every morphism $A \xrightarrow{f} \mathfrak{p}X$ in $\mathcal{C}$ has a cartesian lift in $\mathcal{X}$.

Example

$\mathrm{cod} : \mathcal{C}^{\downarrow} \to \mathcal{C}$ is a fibration iff $\mathcal{C}$ has all pullbacks.

Non-example

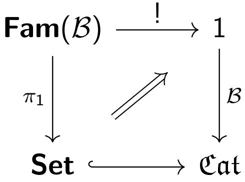
A simple functor which fails to be a fibration: consider category $\mathcal{C}$ consisting of two objects x_0 and x_1 with their identities and an arrow θ_x between them. Let $\mathcal{X}$ be the category extended by a fresh arrow $e \colon x_0 \to x_0$ with $e \circ e = e$ and $\theta_x \circ e = \theta_x$. The functor $\mathcal{X} \to \mathcal{C}$ which sends θ_x to itself and e to id_{x_0} is not a fibration since θ_x in $\mathcal{C}$ does not have a cartesian lift.

More examples of fibrations

- For a covering map $p : E \to B$ of topological spaces the fundamental groupoid functor $\Pi(p) : \Pi(E) \to \Pi(B)$ is a discrete fibration.

More examples of fibrations

- For a covering map $p : E \to B$ of topological spaces the fundamental groupoid functor $\Pi(p) : \Pi(E) \to \Pi(B)$ is a discrete fibration.
- For a category $\mathcal{B}$, the category of families in $\mathcal{B}$ can be constructed as a comma object in $\mathfrak{Cat}$. The projection functor $\pi_1 : \mathbf{Fam}(\mathcal{B}) \to \mathbf{Set}$ is a Grothendieck fibration.



A 2-category of Grothendieck fibrations

Definition

A **fibration map** between two fibrations $\mathfrak{q} : \mathcal{Y} \to \mathcal{D}$ and $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ consists of a pair functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{Y} \to \mathcal{X}$ such that

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{G} & \mathcal{X} \\ {\scriptstyle \mathfrak{q}}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}} \\ \mathcal{D} & \xrightarrow[F]{} & \mathcal{C} \end{array}$$

commutes, and moreover, G is cartesian, that is it carries $\mathfrak{q}$-cartesian morphisms to $\mathfrak{p}$- cartesian morphisms. A **fibration transformation** is a pair of natural transformations $(\alpha\colon F_0 \to F_1, \beta\colon G_0 \to G_1)$ such that $\mathfrak{p} \centerdot \beta = \alpha \centerdot \mathfrak{q}$.

A 2-category of Grothendieck fibrations

Definition

A **fibration map** between two fibrations $\mathfrak{q} : \mathcal{Y} \to \mathcal{D}$ and $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ consists of a pair functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{Y} \to \mathcal{X}$ such that

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{G} & \mathcal{X} \\ {\scriptstyle \mathfrak{q}}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}} \\ \mathcal{D} & \xrightarrow[F]{} & \mathcal{C} \end{array}$$

commutes, and moreover, G is cartesian, that is it carries $\mathfrak{q}$-cartesian morphisms to $\mathfrak{p}$- cartesian morphisms. A **fibration transformation** is a pair of natural transformations $(\alpha \colon F_0 \to F_1, \beta \colon G_0 \to G_1)$ such that $\mathfrak{p} \centerdot \beta = \alpha \centerdot \mathfrak{q}$.

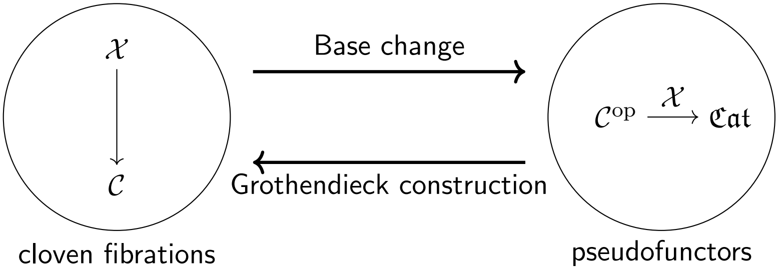
Fibrations, fibration maps, and fibration transformations form a 2-category.

A 2-category of Grothendieck fibrations

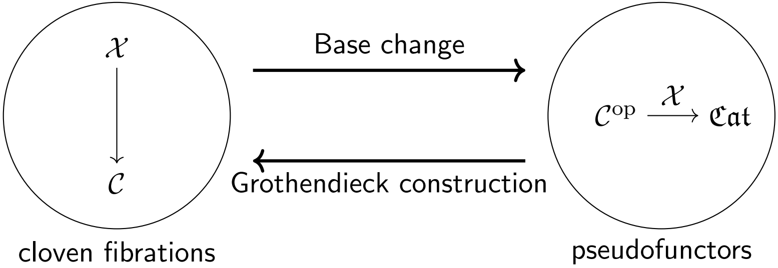
By taking fibres of a cloven fibration over $\mathcal{C}$ we get a $\mathcal{C}$-indexed category. Grothendieck construction makes this into a biequivalence of 2-categories:

A 2-category of Grothendieck fibrations

By taking fibres of a cloven fibration over $\mathcal{C}$ we get a $\mathcal{C}$-indexed category. Grothendieck construction makes this into a biequivalence of 2-categories:



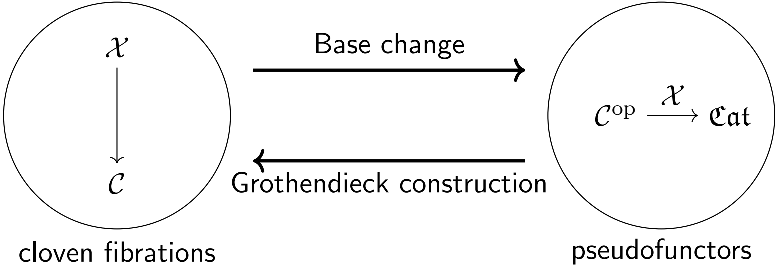
A 2-category of Grothendieck fibrations



$$\mathbf{clFib}(\mathcal{C}) \simeq \mathbf{PsFun}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat})$$

where **clFib** is the 2-category of cloven fibrations.

A 2-category of Grothendieck fibrations



Remark

An interesting feature of the Grothendieck construction is that it reduces category level. That is it turns a 2-functor of 2-categories into a single morphisms in 2-category of categories. Other than a change in viewpoint it makes a world of difference when we work in higher levels.

Fam as example of Grothendieck construction

For a category $\mathcal{B}$, the Grothendieck fibration $\mathbf{Fam}(\mathcal{B}) \to \mathbf{Set}$ is the Grothendieck construction of 2-functor

$$\mathbf{Fun}(-, \mathcal{B}) \colon \mathbf{Set}^{\mathrm{op}} \to \mathfrak{Cat}$$

where for an (indexing) set I, $\mathbf{Fun}(I, \mathcal{B})$ is the category of functors from discrete category I to category $\mathcal{B}$.

Going one dimension higher

2-cartesian 1-cells

Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor of (strict) 2-categories.

Going one dimension higher

2-cartesian 1-cells

Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor of (strict) 2-categories.

Definition

A 1-cell $u\colon X \to Y$ in $\mathbb{X}$ is **cartesian** with respect to $\mathbb{P}$ whenever for each 0-cell W in $\mathbb{X}$ the following commuting square is a (strict) pullback diagram in 2-category $\mathfrak{Cat}$.

$$\begin{array}{ccc} \mathbb{X}(W,X) & \xrightarrow{u_*} & \mathbb{X}(W,Y) \\ \downarrow{\scriptstyle \mathbb{P}_{W,X}} & \lrcorner & \downarrow{\scriptstyle \mathbb{P}_{W,Y}} \\ \mathbb{C}(\mathbb{P}W,\mathbb{P}X) & \xrightarrow[\mathbb{P}(u)_*]{} & \mathbb{C}(\mathbb{P}W,\mathbb{P}Y) \end{array}$$

Going one dimension higher

2-cartesian 1-cells

Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor of (strict) 2-categories.

Definition

A 1-cell $u\colon X \to Y$ in $\mathbb{X}$ is **cartesian** with respect to $\mathbb{P}$ whenever for each 0-cell W in $\mathbb{X}$ the following commuting square is a (strict) pullback diagram in 2-category $\mathfrak{Cat}$.

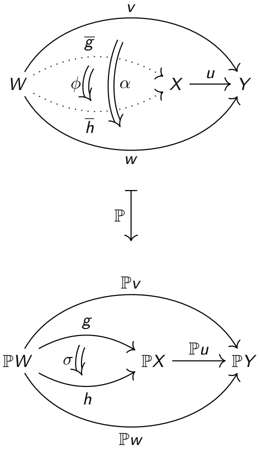
$$\begin{array}{ccc} \mathbb{X}(W,X) & \xrightarrow{u_*} & \mathbb{X}(W,Y) \\ \Big\downarrow{\scriptstyle \mathbb{P}_{W,X}} & \lrcorner & \Big\downarrow{\scriptstyle \mathbb{P}_{W,Y}} \\ \mathbb{C}(\mathbb{P}W,\mathbb{P}X) & \xrightarrow[\mathbb{P}(u)_*]{} & \mathbb{C}(\mathbb{P}W,\mathbb{P}Y) \end{array}$$

Remark

By considering object component of pullback diagram above we observe that every 2-cartesian 1-cell is automatically 1-cartesian in the usual sense.

2-cartesian 1-cells in elementary terms

This definition gives us two layers of cartesian properties of 1-cells w.r.t. $\mathbb{P}$ in $\mathbb{X}$. First of all, u is 1-cartesian as usual. Second, every 2-cell $\alpha\colon v \Rightarrow w\colon W \to Y$ and every 2-cell $\sigma\colon g \Rightarrow h\colon \mathbb{P}W \to \mathbb{P}X$ with $\mathbb{P}(\alpha) = \mathbb{P}(u) \centerdot \sigma$ there is a unique lift ϕ of σ such that $u \centerdot \phi = \alpha$.



Strict 2-fibrations

Definition

A 2-functor $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a **2-fibration** if

1. any 1-cell in $\mathbb{C}$ of the form $f\colon A \to \mathbb{P}X$ has a 2-cartesian lift,

Strict 2-fibrations

Definition

A 2-functor $\mathbb{P} \colon \mathbb{X} \to \mathbb{C}$ is a **2-fibration** if

1. any 1-cell in $\mathbb{C}$ of the form $f \colon A \to \mathbb{P}X$ has a 2-cartesian lift,
2. $\mathbb{P}$ is a local fibration, that is for any pair of objects X, Y in $\mathbb{X}$, the functor $\mathbb{P}_{X,Y} \colon \mathbb{X}(X, Y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a Grothendieck fibration,

Strict 2-fibrations

Definition

A 2-functor $\mathbb{P} \colon \mathbb{X} \to \mathbb{C}$ is a **2-fibration** if

1. any 1-cell in $\mathbb{C}$ of the form $f \colon A \to \mathbb{P}X$ has a 2-cartesian lift,
2. $\mathbb{P}$ is a local fibration, that is for any pair of objects X, Y in $\mathbb{X}$, the functor $\mathbb{P}_{X,Y} \colon \mathbb{X}(X, Y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a Grothendieck fibration,
3. cartesian 2-cells in $\mathbb{X}$ are closed under pre-composition and post-composition with arbitrary 1-cells.

Strict 2-fibrations

Definition

A 2-functor $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a **2-fibration** if

1. any 1-cell in $\mathbb{C}$ of the form $f\colon A \to \mathbb{P}X$ has a 2-cartesian lift,
2. $\mathbb{P}$ is a local fibration, that is for any pair of objects X, Y in $\mathbb{X}$, the functor $\mathbb{P}_{X,Y}\colon \mathbb{X}(X,Y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a Grothendieck fibration,
3. cartesian 2-cells in $\mathbb{X}$ are closed under pre-composition and post-composition with arbitrary 1-cells.

Strict 2-fibrations

Definition

A 2-functor $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a **2-fibration** if

1. any 1-cell in $\mathbb{C}$ of the form $f\colon A \to \mathbb{P}X$ has a 2-cartesian lift,
2. $\mathbb{P}$ is a local fibration, that is for any pair of objects X, Y in $\mathbb{X}$, the functor $\mathbb{P}_{X,Y}\colon \mathbb{X}(X,Y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a Grothendieck fibration,
3. cartesian 2-cells in $\mathbb{X}$ are closed under pre-composition and post-composition with arbitrary 1-cells.

Remark

The second condition is equivalent to say that for any morphism g in $\mathbb{X}$ and a 2-cell $\alpha\colon f \Rightarrow \mathbb{P}g$, there is a cartesian 2-cell $\sigma\colon f \Rightarrow g$ with $\mathbb{P}\sigma = \alpha$.

An archetypal example of strict 2-fibration

Example

The 2-category **Fib** of Grothendieck fibrations is a 2-fibred over 2-category of categories via the codomain functor $\mathrm{cod} \colon \mathbf{Fib} \to \mathfrak{Cat}$

An archetypal example of strict 2-fibration

Example

The 2-category **Fib** of Grothendieck fibrations is a 2-fibred over 2-category of categories via the codomain functor $\text{cod}\colon \mathbf{Fib} \to \mathfrak{Cat}$

To my knowledge this was fist proved and written down explicitly in Claudio Hermida (1999). "Some properties of Fib as a fibred 2-category". In: vol. 134, pp. 83–109.

Weak cartesian 1-cells

Definition

Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor. A 1-cell $f\colon X \to Y$ in $\mathbb{X}$ is **weakly cartesian** with respect to $\mathbb{P}$ whenever for each 0-cell W in $\mathbb{X}$ the following commuting square is a bipullback diagram in 2-category $\mathfrak{Cat}$ of categories.

$$\begin{array}{ccc} \mathbb{X}(W,X) & \xrightarrow{f_*} & \mathbb{X}(W,Y) \\ {\scriptstyle \mathbb{P}_{W,X}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathbb{P}_{W,Y}} \\ \mathbb{C}(\mathbb{P}W,\mathbb{P}X) & \xrightarrow[\mathbb{P}(f)_*]{} & \mathbb{C}(\mathbb{P}W,\mathbb{P}Y) \end{array}$$

Weak cartesian 1-cells

Definition

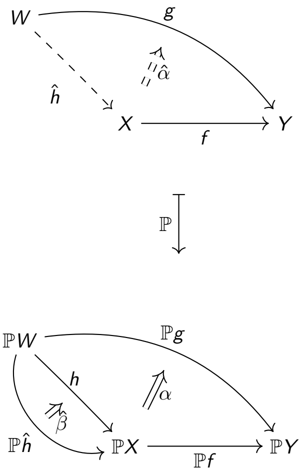
Suppose $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ is a 2-functor. A 1-cell $f\colon X \to Y$ in $\mathbb{X}$ is **weakly cartesian** with respect to $\mathbb{P}$ whenever for each 0-cell W in $\mathbb{X}$ the following commuting square is a bipullback diagram in 2-category $\mathfrak{Cat}$ of categories.

$$\begin{array}{ccc} \mathbb{X}(W,X) & \xrightarrow{f_*} & \mathbb{X}(W,Y) \\ \downarrow{\scriptstyle \mathbb{P}_{W,X}} & \lrcorner & \downarrow{\scriptstyle \mathbb{P}_{W,Y}} \\ \mathbb{C}(\mathbb{P}W,\mathbb{P}X) & \xrightarrow[\mathbb{P}(f)_*]{} & \mathbb{C}(\mathbb{P}W,\mathbb{P}Y) \end{array}$$

Igor Bakovic (2012). "Fibrations in tricategories". In: *93rd Peripatetic Seminar on Sheaves and Logic, University of Cambridge* and
Mitchell Buckley (2014). "Fibred 2-categories and bicategories". In: vol. 218, pp. 1034–1074

Weak cartesian 1-cells in elementary terms

Only a bit more complicated than last one-lifts up to iso



Input data:

1. $g\colon W \to Y$
2. $h\colon \mathbb{P}W \to \mathbb{P}X$
3. iso 2-cell $\alpha\colon \mathbb{P}(f) \circ h \Rightarrow \mathbb{P}(g)$

Output data:

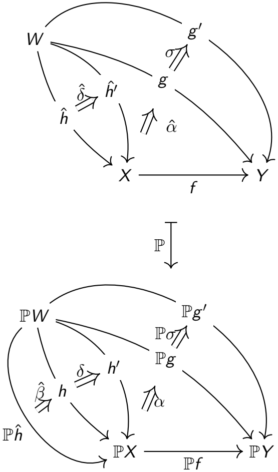
(not necc. unique)

1. $\hat{h}\colon W \to X$
2. iso 2-cell $\hat{\alpha}\colon f\hat{h} \Rightarrow g$
3. iso 2-cell $\hat{\beta}\colon \mathbb{P}(\hat{h}) \Rightarrow h$
4. an equality of 2-cells

$\alpha \circ (\mathbb{P}(f) \centerdot \hat{\beta}) = \mathbb{P}(\hat{\alpha}) \circ \Phi_{h,f}$

Weak 2-cartesian 1-cells in elementary terms

Continued



Input data:

1. $\sigma \colon g \Rightarrow g' \colon W \rightrightarrows Y$
2. $\delta \colon h \Rightarrow h' \colon \mathbb{P}W \rightrightarrows \mathbb{P}X$
3. iso 2-cells
   $\alpha \colon \mathbb{P}(f) \circ h \Rightarrow \mathbb{P}(g)$
   $\alpha' \colon \mathbb{P}(f) \circ h' \Rightarrow \mathbb{P}(g)$
4. an equality of 2-cells
   $\alpha' \circ (\mathbb{P}f \centerdot \delta) = \mathbb{P}(\sigma) \circ \alpha$

Output data:

1. unique $\hat{\delta} \colon \hat{h} \Rightarrow \hat{h}'$
2. an equality $\hat{\alpha}' \circ (f \centerdot \hat{\delta}) = \sigma \circ \hat{\alpha}$
3. an equality $\delta \centerdot (\hat{\beta}) = \hat{\beta}' \circ \mathbb{P}\hat{\delta}$

Cartesian 2-cells

Definition

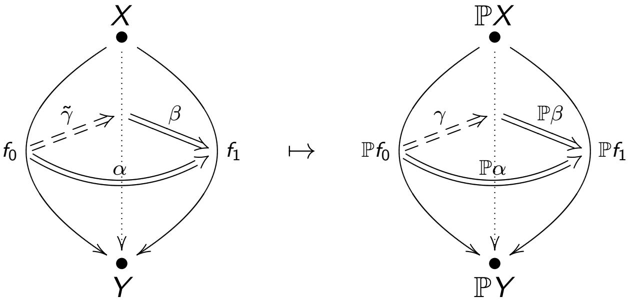
A 2-cell $\alpha : f \Rightarrow g : x \to y$ in $\mathbb{X}$ is **cartesian** if it is cartesian as a 1-cell for the functor $\mathbb{P}_{xy} : \mathbb{X}(x, y) \to \mathbb{C}(\mathbb{P}x, \mathbb{P}y)$.

Cartesian 2-cells

Definition

A 2-cell $\alpha \colon f \Rightarrow g \colon x \to y$ in $\mathbb{X}$ is **cartesian** if it is cartesian as a 1-cell for the functor $\mathbb{P}_{xy} \colon \mathbb{X}(x,y) \to \mathbb{C}(\mathbb{P}x, \mathbb{P}y)$.

In elementary terms it means a 2-cell $\alpha \colon f_0 \Rightarrow f_1 \colon X \rightrightarrows Y$ is cartesian if for any given 1-cell $e \colon X \to Y$ and 2-cell $\beta \colon e \to f_1$ with $\mathbb{P}\alpha = \mathbb{P}\beta \circ \gamma$ for some 2-cell γ, then there is a unique 2-cell $\tilde{\gamma}$ over γ such that $\alpha = \beta \circ \tilde{\gamma}$.



Weak 2-fibrations

As in the case of strict 2-fibrations, we say that $\mathbb{P}$ is *locally fibred* when $\mathbb{P}_{XY} \colon \mathbb{X}(x, y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a fibration for all X, Y in $\mathbb{X}$.

Definition

Let $\mathbb{P} \colon \mathbb{X} \to \mathbb{C}$ be a 2-functor. We say that $\mathbb{P}$ is a **fibration** whenever

1. for any $X \in \mathbb{X}$ and $f \colon B \to \mathbb{P}X$ in $\mathbb{C}$, there is a weakly cartesian 1-cell $\widetilde{f} \colon \widetilde{B} \to X$ with $\mathbb{P}\widetilde{f} = f$;

Weak 2-fibrations

As in the case of strict 2-fibrations, we say that $\mathbb{P}$ is *locally fibred* when $\mathbb{P}_{XY} \colon \mathbb{X}(x, y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a fibration for all X, Y in $\mathbb{X}$.

Definition

Let $\mathbb{P} \colon \mathbb{X} \to \mathbb{C}$ be a 2-functor. We say that $\mathbb{P}$ is a **fibration** whenever

1. for any $X \in \mathbb{X}$ and $f \colon B \to \mathbb{P}X$ in $\mathbb{C}$, there is a weakly cartesian 1-cell $\widetilde{f} \colon \widetilde{B} \to X$ with $\mathbb{P}\widetilde{f} = f$;
2. $\mathbb{P}$ is locally fibred

Weak 2-fibrations

As in the case of strict 2-fibrations, we say that $\mathbb{P}$ is *locally fibred* when $\mathbb{P}_{XY}\colon \mathbb{X}(x,y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a fibration for all X, Y in $\mathbb{X}$.

Definition

Let $\mathbb{P}\colon \mathbb{X} \to \mathbb{C}$ be a 2-functor. We say that $\mathbb{P}$ is a **fibration** whenever

1. for any $X \in \mathbb{X}$ and $f\colon B \to \mathbb{P}X$ in $\mathbb{C}$, there is a weakly cartesian 1-cell $\widetilde{f}\colon \widetilde{B} \to X$ with $\mathbb{P}\widetilde{f} = f$;
2. $\mathbb{P}$ is locally fibred
3. The horizontal composite of any two cartesian 2-cells is again cartesian.

Weak 2-fibrations

As in the case of strict 2-fibrations, we say that $\mathbb{P}$ is *locally fibred* when $\mathbb{P}_{XY} \colon \mathbb{X}(x, y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a fibration for all X, Y in $\mathbb{X}$.

Definition

Let $\mathbb{P} \colon \mathbb{X} \to \mathbb{C}$ be a 2-functor. We say that $\mathbb{P}$ is a **fibration** whenever

1. for any $X \in \mathbb{X}$ and $f \colon B \to \mathbb{P}X$ in $\mathbb{C}$, there is a weakly cartesian 1-cell $\widetilde{f} \colon \widetilde{B} \to X$ with $\mathbb{P}\widetilde{f} = f$;
2. $\mathbb{P}$ is locally fibred
3. The horizontal composite of any two cartesian 2-cells is again cartesian.

Weak 2-fibrations

As in the case of strict 2-fibrations, we say that $\mathbb{P}$ is *locally fibred* when $\mathbb{P}_{XY} \colon \mathbb{X}(x, y) \to \mathbb{C}(\mathbb{P}X, \mathbb{P}Y)$ is a fibration for all X, Y in $\mathbb{X}$.

Definition

Let $\mathbb{P} \colon \mathbb{X} \to \mathbb{C}$ be a 2-functor. We say that $\mathbb{P}$ is a **fibration** whenever

1. for any $X \in \mathbb{X}$ and $f \colon B \to \mathbb{P}X$ in $\mathbb{C}$, there is a weakly cartesian 1-cell $\widetilde{f} \colon \widetilde{B} \to X$ with $\mathbb{P}\widetilde{f} = f$;
2. $\mathbb{P}$ is locally fibred
3. The horizontal composite of any two cartesian 2-cells is again cartesian.

Mitchell Buckley (2014). "Fibred 2-categories and bicategories". In: vol. 218, pp. 1034–1074

Internal fibrations to 2-categories

The first and the most obvious way to internalize definition of Grothendieck fibration in a 2-category is the representable approach. The second approach was developed by Street in (Street, 1974) who introduced two-sided fibrations in 2-categories, followed by two-sided fibrations in bicategories. These fibrations are defined as algebras over certain 2-monads on 2-categories, or hyperdoctrines on bicategories respectively, and Chevalleys internal characterization of fibrations was obtained as a theorem.

Internal fibrations to 2-categories

The third approach was developed by Johnstone in (Johnstone, 1993) which is closer than Streets definition to the spirit of Grothendiecks original definition. For instance, the base change functors is part of data of definition. Johnstone also established the equivalence of his definition with the representable one.

Internal fibrations to 2-categories

Unlike Street's definition, Johnstone's definition does not require strictness of the 2-category nor the existence of the structure of strict pullbacks and comma objects. Indeed, this definition is most suitable for weak 2-categories such as 2-category of toposes where we do not expect diagrams of 1-cells to commute strictly. This definition is also very flexible in terms of existence of bipullbacks: one only needs existence of bipullbacks of the class of 1-cells one would like to define as (op)fibrations. We will call these 1-cells *carrable*.

Internal fibrations to 2-categories

Unlike Street's definition, Johnstone's definition does not require strictness of the 2-category nor the existence of the structure of strict pullbacks and comma objects. Indeed, this definition is most suitable for weak 2-categories such as 2-category of toposes where we do not expect diagrams of 1-cells to commute strictly. This definition is also very flexible in terms of existence of bipullbacks: one only needs existence of bipullbacks of the class of 1-cells one would like to define as (op)fibrations. We will call these 1-cells *carrable*.

Ross Street (1974). "Fibrations and Yoneda's lemma in a 2-category". In: *Lecture Notes in Math., Springer, Berlin* Vol.420, pp. 104–133

Peter Johnstone (1993). "Fibrations and partial products in a 2-category". In: *Applied Categorical Structures* vol.1

Fibration 0-cells in 2-category $\mathbb{K}^{\mathbb{I}}$

Let $p\colon E \to S$ be a carrable 1-cell in $\mathbb{K}$. We call p a **fibration** 0-cell in 2-category $\mathbb{K}^{\mathbb{I}}$ whenever for any 2-cell $\underline{\alpha} : \underline{f} \Rightarrow \underline{g} : A \to S$ in $\mathbb{K}$, we have

- a 1-cell $\langle \overline{r(\alpha)}, \overset{\blacktriangledown}{r_\alpha}, 1_A \rangle \colon \underline{g}^* p \to \underline{f}^* p$
- and a 2-cell $(\overline{\alpha}, \underline{\alpha})\colon f \circ r(\alpha) \Rightarrow g$

Unpack

Fibration 0-cells in 2-category $\mathbb{K}^{\mathbb{I}}$

Let $p \colon E \to S$ be a carrable 1-cell in $\mathbb{K}$. We call p a **fibration** 0-cell in 2-category $\mathbb{K}^{\mathbb{I}}$ whenever for any 2-cell $\underline{\alpha} : \underline{f} \Rightarrow \underline{g} : A \to S$ in $\mathbb{K}$, we have

- a 1-cell $\langle \overline{r(\alpha)}, \overset{\blacktriangledown}{r_\alpha}, 1_A \rangle \colon \underline{g}^* p \to \underline{f}^* p$
- and a 2-cell $(\overline{\alpha}, \underline{\alpha}) \colon f \circ r(\alpha) \Rightarrow g$

in $\mathbb{K}^{\mathbb{I}}$, and moreover the following axioms are satisfied:

Unpack

1. There is an isomorphism $(\overline{\tau_f}, id_{1_A}) : id_{\underline{f}^\star p} \Rightarrow r_{\underline{f}}$ such that $(\overline{id_f}, id_{\underline{f}}) \circ (\overline{f}\,\overline{\tau_0}, id_{\underline{f}}) = (id_{\overline{f}}, id_{\underline{f}})$.

Unpack

Fibration 0-cells in 2-category $\mathbb{K}^{\mathbb{I}}$

Let $p \colon E \to S$ be a carrable 1-cell in $\mathbb{K}$. We call p a **fibration** 0-cell in 2-category $\mathbb{K}^{\mathbb{I}}$ whenever for any 2-cell $\underline{\alpha} : \underline{f} \Rightarrow \underline{g} : A \to S$ in $\mathbb{K}$, we have

- a 1-cell $\langle \overline{r(\alpha)}, \overset{\blacktriangledown}{r}_{\alpha}\ , 1_A \rangle \colon \underline{g}^* p \to \underline{f}^* p$
- and a 2-cell $(\overline{\alpha}, \underline{\alpha}) \colon f \circ r(\alpha) \Rightarrow g$

in $\mathbb{K}^{\mathbb{I}}$, and moreover the following axioms are satisfied:

Unpack

1. There is an isomorphism $(\overline{\tau_f}, id_{1_A}) : id_{\underline{f}^\star p} \Rightarrow r_{\underline{f}}$ such that $(\overline{id_f}, id_{\underline{f}}) \circ (\overline{f}\,\overline{\tau_0}, id_{\underline{f}}) = (id_{\overline{f}}, id_{\underline{f}})$.

Unpack

2. If $\underline{\beta} : \underline{g} \Rightarrow \underline{h}$ is another 2-cell in $\mathbb{K}$, then there exists an iso 2-cell $\tau_{\alpha,\beta} : r(\alpha) \circ r(\beta) \Rightarrow r(\beta\alpha)$ such that the following diagram of 2-cells in $\mathbb{K}^{\mathbb{I}}$ commutes:

$$\begin{array}{ccc} f \circ r(\alpha) \circ r(\beta) & \xRightarrow{\alpha \centerdot r(\beta)} & g \circ r(\beta) \\ {\scriptstyle f \centerdot \tau_{\alpha,\beta}} \Big\Downarrow & = & \Big\Downarrow {\scriptstyle \beta} \\ f \circ r(\beta\alpha) & \xRightarrow[\beta\alpha]{} & h \end{array}$$

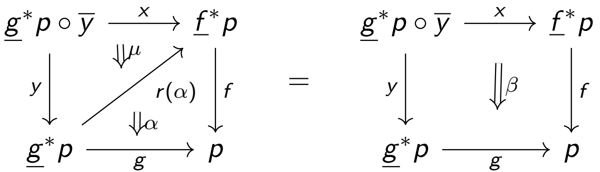
3. Lifting of α is compatible with left whiskering; That is, given any 1-cell $\underline{k} : B \to A$ in $\mathbb{K}$, we require $r(\alpha \centerdot k)$ to fit into the following bipullback square in $\mathbb{K}^{\mathbb{I}}$:

$$\begin{array}{ccc} (\underline{g}\,\underline{k})^* p & \xrightarrow{k_g} & \underline{g}^* p \\ {\scriptstyle r(\alpha.k)}\downarrow & \cong_{\kappa} & \downarrow{\scriptstyle r(\alpha)} \\ (\underline{f}\,\underline{k})^* p & \xrightarrow[k_f]{} & \underline{f}^* p \end{array}$$

where k_f and k_g are pullback 1-cells over $\underline{k}$.
We also require pasting of 2-cells α and κ to be equal to 2-cell $\alpha \centerdot k$.

Unpack

4. For any 1-cells $y = \langle \overline{y}, id, 1_A \rangle$ where $\overline{y}\colon D \to \underline{g}^* E$, and $x = \langle \overline{x}, x, 1_A \rangle \colon \underline{g}^* p \circ \overline{y} \to \underline{f}^* p$ where $\overline{x}\colon D \to \underline{f}^* E$, and , any 2-cell $\beta = \langle \overline{\beta}, \underline{\alpha} \rangle \colon f \circ x \Rightarrow g \circ y$ in $\mathbb{K}^{\mathbb{I}}$ is uniquely factored through α, that is there is a unique 2-cell μ in $\mathbb{K}^{\mathbb{I}}$ with property $(\alpha \centerdot y) \circ (f \centerdot \mu) = \beta$, that is to say the two pasting diagrams in below are equal:



Unpack

Turning iso 2-cells of $\mathbb{K}$ into 1-cells

Suppose $\mathbb{K}$ is a 2-category and $\mathbb{I}$ is the interval category. We can form a new 2-category $\mathbb{K}^{\mathbb{I}} := \mathbf{Fun}_{ps}(\mathbb{I}, \mathbb{K})$ consisting of (strict) 2-functors, pseudo-natural transformations and modifications between them.

Turning iso 2-cells of $\mathbb{K}$ into 1-cells

Suppose $\mathbb{K}$ is a 2-category and $\mathbb{I}$ is the interval category. We can form a new 2-category $\mathbb{K}^{\mathbb{I}} := \mathbf{Fun}_{ps}(\mathbb{I}, \mathbb{K})$ consisting of (strict) 2-functors, pseudo-natural transformations and modifications between them. More explicitly,

- 0-cells are of the form

$$\begin{array}{c} E \\ {\scriptstyle p}\big\downarrow \\ S \end{array}$$

  where $p \in \mathbb{K}_1$.

Turning iso 2-cells of $\mathbb{K}$ into 1-cells

Suppose $\mathbb{K}$ is a 2-category and $\mathbb{I}$ is the interval category. We can form a new 2-category $\mathbb{K}^{\mathbb{I}} := \mathbf{Fun}_{ps}(\mathbb{I}, \mathbb{K})$ consisting of (strict) 2-functors, pseudo-natural transformations and modifications between them. More explicitly,

- 0-cells are of the form

$$\begin{array}{c} E \\ {\scriptstyle p}\big\downarrow \\ S \end{array}$$

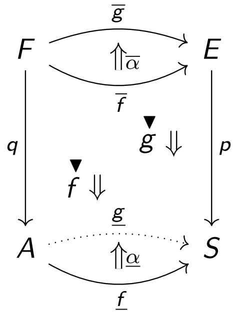
  where $p \in \mathbb{K}_1$.

- 1-cells from q to p are of the form $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$

$$\begin{array}{ccc} F & \xrightarrow{\overline{f}} & E \\ {\scriptstyle q}\big\downarrow & \overset{\blacktriangledown}{f} \Downarrow & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow[\underline{f}]{} & S \end{array}$$

  where $\overset{\blacktriangledown}{f} : p\overline{f} \Rightarrow \underline{f}q$ is an iso 2-cell in $\mathbb{K}$.

- 2-cells between 1-cells f and g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} : \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$ are 2-cells in $\mathbb{K}$



in such a way that the obvious diagram of 2-cells commutes.

Fibrations in 2-cats vs. fibrations between 2-cats

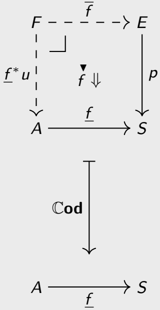
There is a (strict) 2-functor $\mathbb{C}\mathbf{od} \colon \mathbb{K}^{\mathbb{I}} \to \mathbb{K}$ which takes 1-cell p (as in above) to its codomain $\mathbb{C}\mathbf{od}(p)$, a 1-cell f to $\underline{f}$ and a 2-cell $\langle \overline{\alpha}, \underline{\alpha} \rangle$ to $\underline{\alpha}$.

Fibrations in 2-cats vs. fibrations between 2-cats

There is a (strict) 2-functor $\mathbb{C}\mathbf{od} \colon \mathbb{K}^{\mathbb{I}} \to \mathbb{K}$ which takes 1-cell p (as in above) to its codomain $\mathbb{C}\mathbf{od}(p)$, a 1-cell f to $\underline{f}$ and a 2-cell $\langle \overline{\alpha}, \underline{\alpha} \rangle$ to $\underline{\alpha}$.

Proposition

a 1-cell in $\mathbb{K}^{\mathbb{I}}$ is $\mathbb{C}\mathbf{od}$-cartesian iff it is a bipullback square in $\mathbb{K}$.



Proposition

A 1-cell $p\colon E \to S$ in $\mathbb{K}$ is a fibration in the sense of Johnstone iff

- every $\underline{f}\colon A \to \mathbb{C}\mathbf{od}(p)$ has a weakly cartesian lift,
- for every 0-cell q, the 2-functor

$$\mathbb{C}\mathbf{od}_{q,p}\colon \mathbb{K}^{\mathbb{I}}(q,p) \to \mathbb{K}(\mathbb{C}\mathbf{od}(q), \mathbb{C}\mathbf{od}(p))$$

  is a Grothendieck fibration of categories,
- whiskering on the left preserves cartesian 2-cells in $\mathbb{K}^{\mathbb{I}}$.

- Fix an elementary topos $\mathcal{S}$. Every context $\mathbb{T}$ gives rise to an indexed category over $\underline{\mathbb{T}} : \mathfrak{BTop}/\mathcal{S}$, where

$$\underline{\mathbb{T}}(\mathcal{E}) := \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E}) = \text{category of models of } \mathbb{T} \text{ in } \mathcal{E}$$

- Fix an elementary topos $\mathcal{S}$. Every context $\mathbb{T}$ gives rise to an indexed category over $\underline{\mathbb{T}} : \mathfrak{BTop}/\mathcal{S}$, where

$$\underline{\mathbb{T}}(\mathcal{E}) := \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E}) = \text{category of models of } \mathbb{T} \text{ in } \mathcal{E}$$

- Note that $\underline{\mathbb{T}}$ encapsulates data of all the models in all Grothendieck toposes (with base $\mathcal{S}$). (Vickers, 2017) calls them "elephant theories" after (Johnstone, 2002), and also to convey their big structure.

- Fix an elementary topos $\mathcal{S}$. Every context $\mathbb{T}$ gives rise to an indexed category over $\underline{\mathbb{T}} : \mathfrak{BTop}/\mathcal{S}$, where

$$\underline{\mathbb{T}}(\mathcal{E}) :\, = \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E}) = \text{category of models of } \mathbb{T} \text{ in } \mathcal{E}$$

- Note that $\underline{\mathbb{T}}$ encapsulates data of all the models in all Grothendieck toposes (with base $\mathcal{S}$). (Vickers, 2017) calls them "elephant theories" after (Johnstone, 2002), and also to convey their big structure.
- Of course not all elephant theories arise from contexts. For instance if U is a context extension and M is a strict model of context $\mathbb{T}$ in base topos $\mathcal{S}$, then $\mathbb{T}_1/M$ is an elephant theory but not a context.

$$\underline{\mathbb{T}_1/M}(\mathcal{E}) :\, = \text{strict models of } \mathbb{T}_1 \text{ in } \mathcal{E} \text{ which reduce to } p^{\star}M \text{ via } U$$

- Fix an elementary topos $\mathcal{S}$. Every context $\mathbb{T}$ gives rise to an indexed category over $\underline{\mathbb{T}} : \mathfrak{BTop}/\mathcal{S}$, where

$$\underline{\mathbb{T}}(\mathcal{E}) := \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E}) = \text{category of models of } \mathbb{T} \text{ in } \mathcal{E}$$

- Note that $\underline{\mathbb{T}}$ encapsulates data of all the models in all Grothendieck toposes (with base $\mathcal{S}$). (Vickers, 2017) calls them "elephant theories" after (Johnstone, 2002), and also to convey their big structure.
- Of course not all elephant theories arise from contexts. For instance if U is a context extension and M is a strict model of context $\mathbb{T}$ in base topos $\mathcal{S}$, then $\mathbb{T}_1/M$ is an elephant theory but not a context.

$$\underline{\mathbb{T}_1/M}(\mathcal{E}) := \text{strict models of } \mathbb{T}_1 \text{ in } \mathcal{E} \text{ which reduce to } p^\star M \text{ via } U$$

- Certain elephant theories are geometric and have classifying toposes. $\underline{\mathbb{T}}$ and $\underline{\mathbb{T}_1/M}$ are such examples.

Theorem

Suppose $U : \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension. For any model M of $\mathbb{T}_0$ in a (base) topos $\mathcal{S}$, $\mathcal{S}[\mathbb{T}_1/M]$ is an $\mathcal{S}$-topos, and moreover, for any geometric (not necessarily bounded) morphism $\underline{f} : \mathcal{A} \to \mathcal{S}$, the classifying topos $\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M]$ is got by bi-pullback of $\mathcal{S}[\mathbb{T}_1/M]$ along $\underline{f}$:

$$\begin{array}{ccc} \mathcal{A}[\mathbb{T}_1/\underline{f}^\star M] & \xrightarrow{\overline{f}} & \mathcal{S}[\mathbb{T}_1/M] \\ {\scriptstyle p_f}\downarrow & \blacktriangledown\; f \Downarrow & \downarrow{\scriptstyle p} \\ \mathcal{A} & \xrightarrow[\underline{f}]{} & \mathcal{S} \end{array}$$

Theorem

Suppose $U : \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension. For any model M of $\mathbb{T}_0$ in a (base) topos $\mathcal{S}$, $\mathcal{S}[\mathbb{T}_1/M]$ is an $\mathcal{S}$-topos, and moreover, for any geometric (not necessarily bounded) morphism $\underline{f} : \mathcal{A} \to \mathcal{S}$, the classifying topos $\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M]$ is got by bi-pullback of $\mathcal{S}[\mathbb{T}_1/M]$ along $\underline{f}$:

$$
\begin{array}{ccc}
\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M] & \xrightarrow{\;\overline{f}\;} & \mathcal{S}[\mathbb{T}_1/M] \\
{\scriptstyle p_f}\big\downarrow & \overset{\blacktriangledown}{f} \Downarrow & \big\downarrow{\scriptstyle p} \\
\mathcal{A} & \xrightarrow[\;\underline{f}\;]{} & \mathcal{S}
\end{array}
$$

Theorem

If $U : \mathbb{T}_1 \to \mathbb{T}_0$ is an extension map of contexts with fibration property, and M is any model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, then $p : \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is a fibration in the 2-category $\mathfrak{Top}$.

Local homeomorphism of toposes as opfibration

Definition

A geometric morphism $\mathcal{F} \to \mathcal{E}$ is a local homeomorphism whenever $\mathcal{F} \simeq \mathcal{E}/A$ for some object A of $\mathcal{E}$.

Local homeomorphism of toposes as opfibration

Definition

A geometric morphism $\mathcal{F} \to \mathcal{E}$ is a local homeomorphism whenever $\mathcal{F} \simeq \mathcal{E}/A$ for some object A of $\mathcal{E}$.

- For $\mathcal{S}$ a bounded $\mathcal{S}_0$ topos, and $\mathbb{T}_0 = \mathbb{O}$ and $\mathbb{T}_1$ the extended context of $\mathbb{T}_0$ with a fresh edge from terminal to the unique node of $\mathbb{T}_0$.

Local homeomorphism of toposes as opfibration

Definition

A geometric morphism $\mathcal{F} \to \mathcal{E}$ is a local homeomorphism whenever $\mathcal{F} \simeq \mathcal{E}/A$ for some object A of $\mathcal{E}$.

- For $\mathcal{S}$ a bounded $\mathcal{S}_0$ topos, and $\mathbb{T}_0 = \mathbb{O}$ and $\mathbb{T}_1$ the extended context of $\mathbb{T}_0$ with a fresh edge from terminal to the unique node of $\mathbb{T}_0$.
- We get a context extension map $\mathbb{T}_1 \to \mathbb{T}_0$. which is an opfibration.

Local homeomorphism of toposes as opfibration

Definition

A geometric morphism $\mathcal{F} \to \mathcal{E}$ is a local homeomorphism whenever $\mathcal{F} \simeq \mathcal{E}/A$ for some object A of $\mathcal{E}$.

- For $\mathcal{S}$ a bounded $\mathcal{S}_0$ topos, and $\mathbb{T}_0 = \mathbb{O}$ and $\mathbb{T}_1$ the extended context of $\mathbb{T}_0$ with a fresh edge from terminal to the unique node of $\mathbb{T}_0$.
- We get a context extension map $\mathbb{T}_1 \to \mathbb{T}_0$. which is an opfibration.
- And a bipullback of toposes

$$
\begin{array}{ccc}
\mathcal{S}/M \simeq \mathcal{S}[\mathbb{T}_1/M] & \longrightarrow & \mathcal{S}_0[X, x] = \mathcal{S}_0[X][\mathbb{T}_1/X] \\
\downarrow {\scriptstyle M^* p} & & \downarrow {\scriptstyle p} \\
\mathcal{S} & \xrightarrow[M]{} & \mathcal{S}_0[X]
\end{array}
$$

References

- Bakovic, Igor (2012). "Fibrations in tricategories". In: *93rd Peripatetic Seminar on Sheaves and Logic, University of Cambridge*.
- Buckley, Mitchell (2014). "Fibred 2-categories and bicategories". In: vol. 218, pp. 1034–1074.
- Hermida, Claudio (1999). "Some properties of Fib as a fibred 2-category". In: vol. 134, pp. 83–109.
- Johnstone, Peter (1993). "Fibrations and partial products in a 2-category". In: *Applied Categorical Structures* vol.1.
- – (2002). "Sketches of an elephant: A topos theory compendium". In: *Oxford Logic Guides* Vol.1.no.44, Oxford University Press.
- Street, Ross (1974). "Fibrations and Yoneda's lemma in a 2-category". In: *Lecture Notes in Math., Springer, Berlin* Vol.420, pp. 104–133.
- Vickers, Steven (2017). "Arithmetic universes and classifying toposes". In: URL: https://arxiv.org/abs/1701.04611.

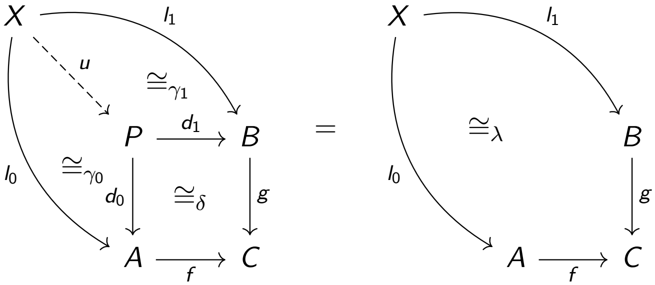
End!

Thank you for your attention!

Bi-pullback: review

A bi-pullback of an opspan $A \xrightarrow{f} C \xleftarrow{g} B$ in a 2-category $\mathbb{K}$ is given by a 0-cell P together with 1-cells d_0, d_1 and an iso 2-cell $\delta \colon fd_0 \Rightarrow gd_1$ satisfying a universal property which states that given another iso cone $(l_0, l_1, \lambda \colon fl_0 \cong gl_1)$ over f, g (with vertex X) there exists a 1-cell u with two iso 2-cells γ_0 and γ_1 such that the pasting diagrams below are equal

Bi-pullback: review



Bi-pullback: review

and furthermore, given 1-cells $u, v \colon X \rightrightarrows P$ and 2-cells $\alpha : d_0 u \Rightarrow d_0 v$ and $\beta : d_1 u \Rightarrow d_1 v$ in such a way that

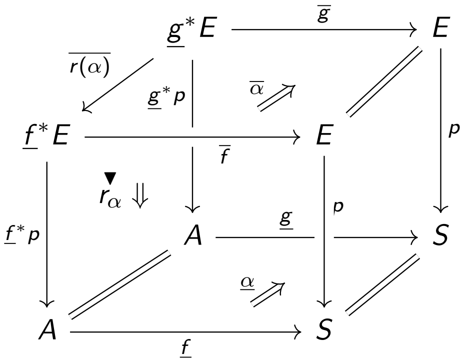
$$\begin{array}{ccc} fd_0 u & \xrightarrow{f.\alpha} & fd_0 v \\ {\scriptstyle \delta.u}\downarrow & & \downarrow{\scriptstyle \delta.v} \\ gd_1 u & \xrightarrow[g.\beta]{} & gd_1 v \end{array}$$

commutes, there exists a unique 2-cell $\sigma \colon u \Rightarrow v$ such that $d_0 \centerdot \sigma = \alpha$ and $d_1 \centerdot \sigma = \beta$.

back to presentation

Supplemental diagrams

Unpacking them yields the following diagram in $\mathbb{K}$:

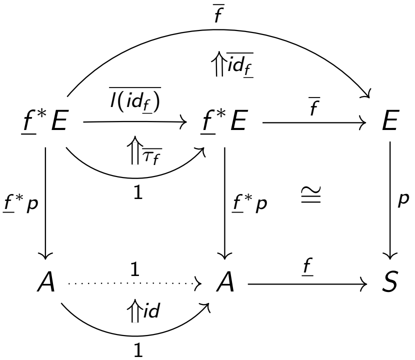


where obvious diagram of 2-cells commutes.

pack

Supplemental diagrams

Unpacking τ_f yields the following diagram in $\mathbb{K}$:



We also get

$$\overset{\blacktriangledown}{r}_{\underline{f}} \circ (\underline{f}^* p \bullet \overline{\tau_f}) = id_{\underline{f}^* p}$$
$$\overline{id_{\underline{f}}} \circ (\overline{f}\,\overline{\tau_f}) = id_{\overline{f}}$$

Supplemental diagrams

Unpacking $\tau_{\alpha,\beta}$ yields the following diagram in $\mathbb{K}$:

$$\begin{array}{ccccc}
\underline{h}^* E & \xrightarrow{\overline{r(\beta)}} & \underline{g}^* E & \xrightarrow{\overline{r(\alpha)}} & \underline{f}^* E \\
\downarrow{\scriptstyle \underline{h}^* p} & \cong & \downarrow{\scriptstyle \underline{g}^* p} & \cong & \downarrow{\scriptstyle \underline{f}^* p} \\
A & \xrightarrow{1} & A & \xrightarrow{1} & A
\end{array}
\qquad \underline{h}^* E \xrightarrow{\overline{r(\beta\circ\alpha)}} \underline{f}^* E \ (\cong)$$

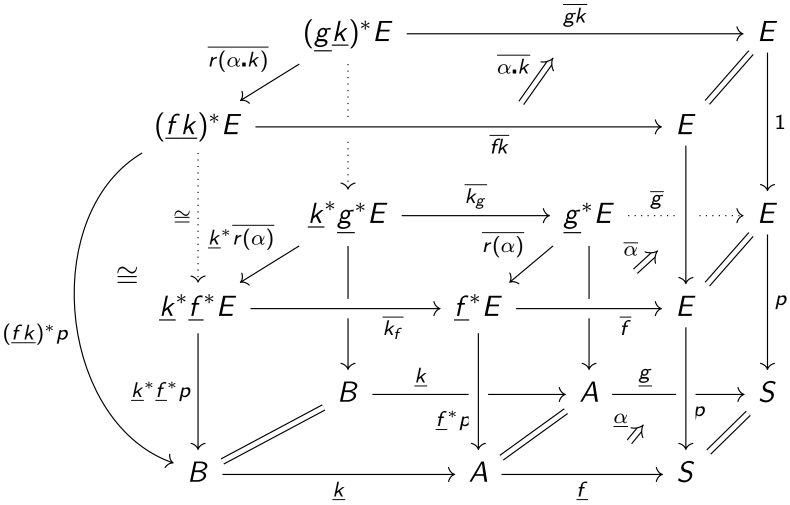
Furthermore, we get

$$\overline{r_{\beta\alpha}} \circ (\overline{f} \cdot \overline{\tau}_{\alpha,\beta}) = \overline{\beta} \circ (\overline{\alpha} \cdot \overline{r(\beta)})$$

$$\overset{\blacktriangledown}{r}_{\beta\alpha} \circ (\underline{f}^* p \cdot \overline{\tau}_{\alpha,\beta}) = \overset{\blacktriangledown}{r}_{\beta} \circ (\overset{\blacktriangledown}{r}_{\alpha} \cdot \overset{\blacktriangledown}{r}_{\beta})$$

pack

$\overline{r(\alpha \centerdot k)}$ is isomorphic to the bi-pullback of $\overline{r(\alpha)}$ along $\overline{k_f}$, which is to say the top left vertical square of the diagram commutes up to an isomorphism.



Supplemental diagrams