Fibration of Toposes

PSSL 101, Leeds

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AUs as finitary approximation of Grothendieck toposes

Pretoposes

1. finite limits
2. stable finite disjoint coproducts
3. stable effective quotients

Giraud's theorem

Arithmetic Universes

1 + 2 + 3

\+ parametrized list objects

$$1 \xrightarrow{\varepsilon} \mathrm{List}(A) \xleftarrow{cons} A \times \mathrm{List}(A)$$

Grothendieck toposes

1 + 2 + 3

\+ set-indexed coproducts

\+ small hom-sets

\+ a set of generators

- There is no reason we should restrict ourselves to Set-toposes. If $\mathcal{S}$ is any elementary topos, any bounded geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is equivalent to $Sh_{\mathcal{S}}(\mathbb{C}, \mathbb{J}) \to \mathcal{S}$ where $(\mathbb{C}, \mathbb{J})$ is an internal site in $\mathcal{S}$ and $Sh_{\mathcal{S}}(\mathbb{C}, \mathbb{J})$ is the topos of $\mathcal{S}$-valued sheaves. This is sometimes known as relativized Giraud's theorem.
- So, we have

A Grothendieck topos over ~~Set~~ $\mathcal{S}$:=
A bounded geometric morphism $p : \mathcal{E} \to \mathcal{S} \simeq$
an internal site in $\mathcal{S}$

More details: [(**Gen.Elephant**) B3.3.4, C2.4]

Note: we always assume base topos $\mathcal{S}$ has n.n.o (same with an AU)

AUs versus Grothendieck toposes

Suppose a geometric theory $\mathbb{T}$ can be expressed in an "arithmetic way".

	AUs	Grothendieck toposes
Classifying space	$\mathbf{AU}\langle\mathbb{T}\rangle$	$\mathcal{S}[\mathbb{T}]$
$\mathbb{T}_2 \to \mathbb{T}_1$	$\mathbf{AU}\langle\mathbb{T}_2\rangle \to \mathbf{AU}\langle\mathbb{T}_1\rangle$	$\mathcal{S}[\mathbb{T}_1] \to \mathcal{S}[\mathbb{T}_2]$
Base	Base independent	Base dependent
Infinities	Intrinsic; provided by List e.g. $N = \mathrm{List}(1)$	Extrinsic; from $\mathcal{S}$ e.g. infinite coproducts in the category of sheaves
Results	A single result in AUs	a family of results for toposes parametrized by base $\mathcal{S}$

Outline of the talk

- Steven Vickers (2016). "Sketches for arithmetic universes". In: URL: https://arxiv.org/abs/1608.01559
  developes a theory of AUs and presents them by a 2-category $\mathfrak{Con}$ of contexts. $\mathfrak{Con}$ has finite strict PIE limits. It also possesses strict pullbacks along certain class of maps called *context extension*.

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- (Op)fibrations in 2-categories:
  - **Fib-Street** internal to representable 2-categories (with strict pullbacks and cotensor with $\mathcal{2}$). Relies on existence of comma objects. Generalizes notion of Grothendieck (op)fibrations in $\mathfrak{Cat}$.
  - Peter Johnstone (1993). "Fibrations and partial products in a 2-category". In: *Applied Categorical Structures* Vol.1, 141–179: internal to 2-categories with bi-pullbacks without use of comma objects in definition.

Outline of the talk

- Using classifying toposes of contexts, we prove that (op)fibrations of contexts give rise to (op)fibrations of toposes. at the level of syntax (contexts) we need strict constructions and the level of semantics we need lax constructions.

$$\left\{\begin{array}{c}\text{Street-Style (op)fibrations}\\ \text{of contexts in } \mathfrak{Con}\\ \text{Syntactic level (strict)}\end{array}\right\} \xrightarrow{\text{models}} \left\{\begin{array}{c}\text{Johnstone-Style (op)fibrations}\\ \text{of toposes in } \mathfrak{Top}\\ \text{Semantic level (non-strict)}\end{array}\right\}$$

2-category $\mathfrak{Con}$

The 2-category $\mathfrak{Con}$ of contexts which is developed in (Vickers, 2016). We start with structure of sketches:
An AU-sketch is a structure with sorts and operations as shown in this diagram.

$$
\begin{tikzcd}
\mathrm{U}^{\mathrm{pb}} \arrow[d, shift right, "\Gamma^1"'] \arrow[d, shift left, "\Gamma^2"] & \mathrm{U}^{\mathrm{list}} \arrow[l, "\Lambda_2"'] \arrow[r, "\Lambda_0"] \arrow[d, shift right, "\mathrm{c}"'] \arrow[d, shift left, "\mathrm{e}"] & \mathrm{U}^1 \arrow[d, "\mathrm{tm}"] \\
\mathrm{G}^2 \arrow[r, "\mathrm{d}_i\ (i=0{,}1{,}2)"] & \mathrm{G}^1 \arrow[r, shift left, "\mathrm{d}_i\ (i=0{,}1)"] & \mathrm{G}^0 \arrow[l, shift left, "\mathrm{s}"] \\
\mathrm{U}^{\mathrm{po}} \arrow[u, shift left, "\Gamma_1"] \arrow[u, shift right, "\Gamma_2"'] & & \mathrm{U}^0 \arrow[u, "\mathrm{i}"']
\end{tikzcd}
$$

- A morphism of AU-sketches is a family of carriers for each sort that also preserves operators. Some of this morphism deserve the name *extension*, which are in fact, finite sequence of simple extensions. A simple extension consist of adding fresh nodes, edges and commutativities for universals which have been freshly added.

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- The next fundamental concept is the notion of *equivalence extension*. When we have a sketch morphism, we may get some derived edges and commutativities. The idea of equivalence extension is to add them at this stage. The added elements are indeed uniquely determined by elements of the original, so the presented AUs are isomorphic as a result of an equivalence extension.

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- The next fundamental concept is the notion of *equivalence extension*. When we have a sketch morphism, we may get some derived edges and commutativities. The idea of equivalence extension is to add them at this stage. The added elements are indeed uniquely determined by elements of the original, so the presented AUs are isomorphic as a result of an equivalence extension.
- *Contexts* are a restricted form of sketches for arithmetic universes. Every 0-cells, 1-cells, and 2-cells in $\mathfrak{Con}$ are introduced in finite number of steps. e.g. $\mathbb{1}$, $\mathbb{O}$, $\mathbb{T}_1 \times \mathbb{T}_2$, $\mathbb{T}^{\rightarrow}$, $\mathbb{T}^{\rightarrow\rightarrow}$, etc.

- Finally, $\mathfrak{Con}(\mathbb{T}_0, \mathbb{T}_1)$ consists of all opspans (E, F) from $\mathbb{T}_0$ to $\mathbb{T}_1$:

$$\mathbb{T}_0 \xrightarrow[\Subset]{E} \mathbb{T}_0' \xleftarrow{F} \mathbb{T}_1$$

  where F is a sketch extension morphism and E an sketch equivalence.
- 2-cells are given as context maps (e, α) from $\mathbb{T}_0$ to $\mathbb{T}_1^{\rightarrow}$ where $\mathbb{T}_0$ and $\mathbb{T}_1$ are themselves contexts.

A central issue for models of sketches is that of *strictness*. The standard sketch-theoretic notion is non-strict: for a universal, such as a pullback of some given opspan, the pullback cone can be interpreted as any pullback of the opspan. Contexts give us good handle over strictness:

Proposition (Vickers, 2017)

Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be an extension map in $\mathfrak{Con}$, that is to say one deriving from an extension $\mathbb{T}_0 \subset \mathbb{T}_1$. Suppose in some AU $\mathcal{A}$ we have a model M_1 of $\mathbb{T}_1$, a strict model M_0' of $\mathbb{T}_0$, and an isomorphism $\phi_0\colon M_0' \cong M_1 U$.

$$\begin{tikzcd} \mathbb{T}_1 \arrow[d, "U"'] \\ \mathbb{T}_0 \end{tikzcd} \qquad \begin{tikzcd} M_1' \arrow[r, dotted, "\phi_1", "\cong"'] \arrow[d, dotted, no head] & M_1 \arrow[d, dotted, no head] \\ M_0' \arrow[r, "\phi_0", "\cong"'] & M_1 U \end{tikzcd}$$

Then there is a unique model M_1' of $\mathbb{T}_1$ and isomorphism $\phi_1\colon M_1' \cong M_1$ such that

1. M_1' is strict,
2. $M_1' U = M_0'$,
3. $\phi_1 U = \phi_0$, and
4. ϕ_1 is equality on all the primitive nodes for the extension $\mathbb{T}_0 \subset \mathbb{T}_1$.

Street-style fibrations in strict 2-categories

- For a representable 2-category $\mathcal{K}$, and for each 0-cell B in $\mathcal{K}$. Street defines a strict KZ-monad on strict slice 2-category $\mathcal{K}/B$. expand
- 1-cells $p : E \to B$ in $\mathcal{K}$ which support the structure a pseudo-algebra w.r.t to this 2-monad are called ***fibrations***.
- 1-cells supporting the structure of an algebra are called ***split*** fibrations.
- For opfibration there is a similar story using another KZ-monad. expand

Street-style fibrations in strict 2-categories

- **Chevalley criteria:**

Proposition (Street)

$p : E \to B$ is a (split) fibration if and only if $\overline{p}$ has a right adjoint with (identity) isomorphism counit.

$$
\begin{tikzcd}
E^{\to} \arrow[d, "e_0"'] \arrow[dr, dotted, "\overline{p}"'] \arrow[drr, bend left, "e_1"] & & \\
E \arrow[dr, bend right, "p"'] & B/p \arrow[ul, dotted, bend right] \arrow[r] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\Phi_p \Uparrow"] & E \arrow[d, "p"] \\
& B \arrow[r, "1"'] & B
\end{tikzcd}
$$

$\mathfrak{Con}$ has pullbacks along context extensions and also cotensor with 2 and that makes possible to use Street's definition to define context fibrations.

Definition

Context extension U is said to be an *extension map with fibration property* if it is a Street-style fibration in the 2-category $\mathfrak{Con}$, that is $\gamma_1 : \mathbb{T}_1^{\rightarrow} \rightarrow \mathrm{cod}^*(\mathbb{T}_1)$ has a right adjoint $\gamma_1 \dashv \lambda_1$ with co-unit of adjunction given by strict equality, that is $\gamma_1 \circ \lambda_1 = id_{\mathrm{cod}^* \mathbb{T}_1}$.

$$
\begin{tikzcd}
\mathbb{T}_1^{\rightarrow} \arrow[rrr, bend right=10, "\gamma_1"'] \arrow[rrr, phantom, "\top"] \arrow[dr, "\gamma_0"'] \arrow[ddd, bend right, "U^{\rightarrow}"'] & & & \mathrm{cod}^*(\mathbb{T}_1) \arrow[lll, dotted, bend right=10, "\lambda_1"'] \arrow[dr, "\iota_c"] \arrow[dd] & \\
& \mathrm{dom}^*(\mathbb{T}_1) \arrow[rrr, bend right=10, "\iota_d"'] \arrow[dd, "U_0"] & & & \mathbb{T}_1 \arrow[dd, "U"] \\
& & & \mathbb{T}_0^{\rightarrow} \arrow[dr, "\mathrm{cod}"] & \\
& \mathbb{T}_0^{\rightarrow} \arrow[urr, equal] \arrow[rrr, bend right=10, "\mathrm{dom}"'] \arrow[rrr, phantom, "\Rightarrow\alpha"] & & & \mathbb{T}_0
\end{tikzcd}
$$

Elephant's Definition of Fibration

- For 2-category of elementary toposes we can not use Street's definition, since this 2-category does not have strict pullbacks, but only bi-pullbacks along bounded geometric morphisms.
- One remedy is to look at Section B.4.4.1 of (**Elephant1**) which provides a definition of fibration for 1-cells in any 2-category with *bi-pullbacks*. However, one only needs existence of bi-pullbacks of the class of 1-cells that one would like to define as fibrations. This definition can be very well used in 2-category of elementary toposes to define *certain bounded geometric morphisms* as fibrations.
- However, Elephant's definition is complicated and difficult to use for purposes of our work. We introduce a 2-category $\mathfrak{GTop}$ and utilize it to simplify Elephant's definition (with slight modification). Essentially, we wrap up the information of iso 2-cells involved in Elephant's definition as part of structure of 1-cells in $\mathfrak{GTop}$.

- We construct a 2-category $\mathfrak{GTop}$ specified by the following data:
- 0-cells are of the form

$$\begin{tikzcd} \mathcal{E} \arrow[d, "p"'] \\ \mathcal{S} \end{tikzcd}$$

$\mathcal{E}$, $\mathcal{S}$: elementary toposes, and p : bounded geometric morphism.

- 1-cells from q to p are of the form $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$, where

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\overline{f}"] \arrow[d, "q"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{A} \arrow[r, "\underline{f}"'] & \mathcal{S} \end{tikzcd}$$

$\overset{\blacktriangledown}{f} : p\overline{f} \Rightarrow \underline{f}q$: isomorphism geometric transformation.

- 2-cells between any two 1-cells f and g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} : \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$ are geometric transformations

$$
\begin{tikzcd}
\mathcal{F} \arrow[rr, bend left, "\overline{g}"] \arrow[rr, bend right, "\overline{f}"'] \arrow[rr, phantom, "\Uparrow\overline{\alpha}"] \arrow[dd, "q"'] & & \mathcal{E} \arrow[dd, "p"] \\
& \overset{\blacktriangledown}{f} \Downarrow \quad \overset{\blacktriangledown}{g} \Downarrow & \\
\mathcal{A} \arrow[rr, bend left, dotted, "\underline{g}"] \arrow[rr, bend right, "\underline{f}"'] \arrow[rr, phantom, "\Uparrow\underline{\alpha}"] & & \mathcal{S}
\end{tikzcd}
$$

in such a way that the obvious diagram of 2-cells commutes.

- Composition of 1-cells $k \colon r \to q$ and $f \colon q \to p$ is given by pasting. more explicitly, $f \circ k = \langle \overline{f} \circ \overline{k}, (\underline{f} \centerdot \overset{\blacktriangledown}{k}) \circ (\overset{\blacktriangledown}{f} \centerdot \overline{k}), \underline{f} \circ \underline{k} \rangle$.
- Vertical and horizontal composition of 2-cells is defined component-wise.
- Identity 1-cells and 2-cells are defined trivially.

Suppose $\mathcal{E}$ and $\mathcal{S}$ are elementary toposes and $p : \mathcal{E} \to \mathcal{S}$ is a bounded geometric morphisms. We call p a fibration in 2-category $\mathfrak{Top}$ whenever for any geometric transformation $\underline{\alpha} : \underline{f} \Rightarrow \underline{g} : \mathcal{A} \to \mathcal{S}$, we have

- a 1-cell $l(\alpha)\colon \underline{g}^* p \to \underline{f}^* p$
- and a 2-cell $\alpha\colon f \circ l(\alpha) \Rightarrow g$

Unpack

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- a 1-cell $I(\alpha) \colon \underline{g}^* p \to \underline{f}^* p$
- and a 2-cell $\alpha \colon f \circ I(\alpha) \Rightarrow g$

in $\mathfrak{GTop}$, and moreover the following axioms are satisfied:

Unpack

1. If $\underline{\alpha} = id_f$, then there exists an isomorphism 2-cell $\iota_0 : id_{\underline{f}^\star p} \Rightarrow I(\alpha)$ in $\mathfrak{GTop}$ with $\underline{\iota_0} = id_{id_A}$ and $\alpha \circ (f \centerdot \tau_0) = id_f$.

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Unpack

2. If $\underline{\beta} : \underline{g} \Rightarrow \underline{h}$ is another geometric transformation, then there exists an isomorphism 2-cell $\iota_{\alpha,\beta} : I(\alpha) \circ I(\beta) \Rightarrow I(\beta\alpha)$ in such a way that the following diagram of 2-cells in $\mathfrak{GTop}$ commutes:

$$\begin{tikzcd} f \circ I(\alpha) \circ I(\beta) \arrow[r, Rightarrow, "\alpha \centerdot I(\beta)"] \arrow[d, Rightarrow, "f \centerdot \iota_{\alpha,\beta}"'] \arrow[dr, phantom, "="] & g \circ I(\beta) \arrow[d, Rightarrow, "\beta"] \\ f \circ I(\beta\alpha) \arrow[r, Rightarrow, "\beta\alpha"'] & h \end{tikzcd}$$

Unpack

3. Lifting of α is compatible with left whiskering; That is, given any geometric morphism $\underline{k} : \mathcal{B} \to \mathcal{A}$ of toposes, we require $l(\alpha \centerdot k)$ to fit into the following bi-pullback square in $\mathfrak{GTop}$:

$$\begin{tikzcd}
(\underline{g}\,\underline{k})^* p \arrow[r, "k_g"] \arrow[d, "l(\alpha . k)"'] \arrow[dr, phantom, "\cong_\kappa"] & \underline{g}^* p \arrow[d, "l(\alpha)"] \\
(\underline{f}\,\underline{k})^* p \arrow[r, "k_f"'] & \underline{f}^* p
\end{tikzcd}$$

where k_f and k_g are pullback 1-cells over $\underline{k}$.
We also require pasting of 2-cells α and κ to be equal to 2-cell $\alpha \centerdot k$.

Unpack

4. For any 1-cells $x = \langle \overline{x}, \overset{\blacktriangledown}{x}, id \rangle$ where $\overline{x} \colon \mathcal{D} \to \underline{f}^{*}\mathcal{E}$, and $y = \langle \overline{y}, id_{(\underline{g}^{*}p)\circ \overline{y}}, id_{\mathcal{A}} \rangle$ where $\overline{y} \colon \mathcal{D} \to \underline{g}^{*}\mathcal{E}$, any 2-cell $\beta = \langle \overline{\beta}, \underline{\alpha} \rangle \colon f \circ x \Rightarrow g \circ y$ in $\mathfrak{GTop}$ is uniquely factored through α, that is there is a unique 2-cell μ in $\mathfrak{GTop}$ with property $(\alpha \centerdot y) \circ (f \centerdot \mu) = \beta$, that is to say the two pasting diagrams in below are equal:

$$
\begin{tikzcd}
\underline{g}^{*}p \circ \overline{y} \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\mu", near start] & \underline{f}^{*}p \arrow[d, "f"] \\
\underline{g}^{*}p \arrow[r, "g"'] \arrow[ur, "l(\alpha)"'] \arrow[ur, phantom, "\Downarrow\alpha", near end, shift right=4] & p
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\underline{g}^{*}p \circ \overline{y} \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\beta"] & \underline{f}^{*}p \arrow[d, "f"] \\
\underline{g}^{*}p \arrow[r, "g"'] & p
\end{tikzcd}
$$

Unpack

- Fix an elementary topos $\mathcal{S}$. Every context $\mathbb{T}$ gives rise to an indexed category over $\underline{\mathbb{T}} : \mathfrak{BTop}/\mathcal{S}$, where

$$\underline{\mathbb{T}}(\mathcal{E}) :\ = \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E}) = \text{category of models of } \mathbb{T} \text{ in } \mathcal{E}$$

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- Note that $\underline{\mathbb{T}}$ encapsulates data of all the models in all Grothendieck toposes (with base $\mathcal{S}$). **AUtoTop** calls them "elephant theories" after (**Gen.Elephant**), and also to convey their big structure.

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- Of course not all elephant theories arise from contexts. For instance if U is a context extension and M is a strict model of context $\mathbb{T}$ in base topos $\mathcal{S}$, then $\mathbb{T}_1/M$ is an elephant theory but not a context.

$$\underline{\mathbb{T}_1/M}(\mathcal{E}) := \text{strict models of } \mathbb{T}_1 \text{ in } \mathcal{E} \text{ which reduce to } p^{\star}M \text{ via } U$$

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- Certain elephant theories are geometric and have classifying toposes. $\underline{\mathbb{T}}$ and $\underline{\mathbb{T}_1/M}$ are such examples.

Theorem (Vickers,2017)

Suppose $U : \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension. For any model M of $\mathbb{T}_0$ in a (base) topos $\mathcal{S}$, $\mathcal{S}[\mathbb{T}_1/M]$ is an $\mathcal{S}$-topos, and moreover, for any geometric (not necessarily bounded) morphism $\underline{f} : \mathcal{A} \to \mathcal{S}$, the classifying topos $\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M]$ is got by bi-pullback of $\mathcal{S}[\mathbb{T}_1/M]$ along $\underline{f}$:

$$\begin{tikzcd}
\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M] \arrow[r, "\overline{f}"] \arrow[d, "p_f"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \mathcal{S}[\mathbb{T}_1/M] \arrow[d, "p"] \\
\mathcal{A} \arrow[r, "\underline{f}"'] & \mathcal{S}
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\mathcal{A} \arrow[r, "\underline{f}"'] & \mathcal{S}
\end{tikzcd}$$

Theorem (S.H. , 2017)

If $U : \mathbb{T}_1 \to \mathbb{T}_0$ is an extension map of contexts with fibration property, and M is any model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, then $p : \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is a fibration in the 2-category $\mathfrak{Top}$.

$$\begin{tikzcd}
 & \mathcal{A}[\mathbb{T}_1/\underline{g}^\star M] \arrow[rr, "\overline{g}"] \arrow[dl, "\overline{I(\alpha)}"'] \arrow[dd, "p_g"] \arrow[dr, phantom, "\overset{\overline{\alpha}}{\Rightarrow}"] & & \mathcal{S}[\mathbb{T}_1/M] \arrow[dl, equal] \arrow[dd, "p"] \\
\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M] \arrow[rr, "\overline{f}"] \arrow[dd, "p_f"'] & & \mathcal{S}[\mathbb{T}_1/M] \arrow[dd, "p"] & \\
 & \mathcal{A} \arrow[rr, "\underline{g}"] \arrow[dl, equal] \arrow[dr, phantom, "\overset{\underline{\alpha}}{\Rightarrow}"] & & \mathcal{S} \arrow[dl, equal] \\
\mathcal{A} \arrow[rr, "\underline{f}"'] & & \mathcal{S} &
\end{tikzcd}$$

- finding $\overline{I(\alpha)}$: equivalent to finding a model of $\mathbb{T}_1/\underline{f}^\star M$ in $\mathcal{A}[\mathbb{T}_1/\underline{g}^\star M]$.

$$\mathfrak{g} := (G^{\mathbb{T}_1}_{\underline{g}^\star M}, {p_g}^\star {\underline{\alpha}_M}^\star) \in \mathrm{cod}^*(\mathbb{T}_1)\text{-}\mathbf{Mod}\text{-}\,\mathcal{A}[\mathbb{T}_1/\underline{g}^\star M]$$

Model $N: \ = \mathfrak{g} \centerdot (\lambda_1; \gamma_0; \delta_d)$ corresponds to $\overline{I(\alpha)}$.

- $\mathfrak{g}\,.\,\lambda_1$: induces a 1-cell $v : u \to p$ in $\mathfrak{GTop}$ with $\underline{v} = id$.
- We have $\theta = (\overline{\theta}, \underline{\alpha})$ and $f \circ l(\alpha) \cong vq_0$ and $v \circ q_1 \cong g$. Pasting all these 2-cells in $\mathfrak{GTop}$ defines $\alpha = (\overline{\alpha}, \underline{\alpha})$.

$$\begin{tikzcd}
p_g \arrow[r, bend left, "q_1"] \arrow[r, bend right, "q_0"'] \arrow[r, phantom, "\Uparrow\theta"] \arrow[rr, bend left=70, "g"] \arrow[dr, bend right, "l(\alpha)"'] & u \arrow[r, "v"] & p \\
 & p_f \arrow[ur, bend right, "f"'] &
\end{tikzcd}$$

Local homeomorphism of toposes as opfibration

Definition

A geometric morphism $\mathcal{F} \to \mathcal{E}$ is a local homeomorphism whenever $\mathcal{F} \simeq \mathcal{E}/A$ for some object A of $\mathcal{E}$.

For $\mathcal{S}$ a bounded $\mathcal{S}_0$ topos, and $\mathbb{T}_0 = \mathbb{O}$ and $\mathbb{T}_1$ the extended context of $\mathbb{T}_0$ with a fresh edge from terminal to the unique node of $\mathbb{T}_0$:

$$\begin{tikzcd}
\mathcal{S}/M \simeq \mathcal{S}[\mathbb{T}_1/M] \arrow[r] \arrow[d, "M^* p"'] & \mathcal{S}_0[X, x] = \mathcal{S}_0[X][\mathbb{T}_1/X] \arrow[d, "p"] \\
\mathcal{S} \arrow[r, "M"'] & \mathcal{S}_0[X]
\end{tikzcd}$$

References

Johnstone, Peter (1993). "Fibrations and partial products in a 2-category". In: *Applied Categorical Structures* Vol.1, 141–179.

Vickers, Steven (2016). "Sketches for arithmetic universes". In: URL: https://arxiv.org/abs/1608.01559.

End!

Thank you for your attention!

A 2-monad of Street

Let $\mathcal{K}$ be a representable 2-category. Define $\mathcal{K}/B$ to be the (strict) slice 2-category over B. (**Fib-Street**) constructs a 2-monad $R : \mathcal{K}/B \to \mathcal{K}/B$ which takes an object (E, p) to $(B/p, R(p))$ where

$$\begin{tikzcd} B/p \arrow[r, "e"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\Phi_p\Uparrow"] & E \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$

is a comma square.

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A 2-monad of Street

Remark

Φ_p can be decomposed as follows:

$$
\begin{tikzcd}
B/p \arrow[r, "e"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\Phi_p\Uparrow"] & E \arrow[d, "p"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}
=
\begin{tikzcd}
B/p \arrow[r, "\hat{d_1}"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Phi\Uparrow"] & B \arrow[d, "1"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}
$$

A 2-monad of Street

If $f : E' \to E$ is a 1-cell in $\mathcal{K}/B$, then define B/f to be the unique 1-cell with $f \circ \hat{d}_1' = \hat{d}_1 \circ B/f$ and $\hat{p} \circ B/f = \hat{p}'$.

$$\begin{tikzcd}
B/p' \arrow[r, "\hat{d}_1'"] \arrow[d, dotted, "B/f"'] \arrow[dd, bend right=60, "\hat{p}'"'] \arrow[dr, phantom, "\lrcorner" very near start] & E' \arrow[d, "f"] \\
B/p \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, "p"] \\
B^{\to} \arrow[r, "d_1"'] & B
\end{tikzcd}$$

A 2-monad of Street

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$$\begin{tikzcd}
B/p' \arrow[r, "\hat{d}_1'"] \arrow[d, dotted, "B/f"'] \arrow[dd, bend right=60, "\hat{p}'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, "f"] \\
B/p \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
B^{\to} \arrow[r, "d_1"'] & B
\end{tikzcd}$$

Similarly if $\sigma : f \Rightarrow g$ is a 2-cell then we have a unique induced 2-cell $B/\sigma \colon B/f \Rightarrow B/g$ with $\hat{d}_1 \circ B/\sigma = \sigma \circ \hat{d}_1'$ and $\hat{p} \circ B/\sigma = id_{\hat{p}'}$.

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Proposition

R is a KZ monad.

A 2-monad of Street

Unit of monad $i : id \Rightarrow R$ at (E, p) is given by the unique arrow $i_p : E \to B/p$ with property that $R(p) \circ i_p = p$ and $\hat{d_1} \circ i_p = 1_E$, and moreover $\Phi_p \centerdot i_p = id_p$, all inferred by universal property of comma object B/p.

$$
\begin{tikzcd}
E \arrow[r, dotted, "i_p"] \arrow[dd, "p"'] & B/p \arrow[r, "\hat{d_1}"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
& B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Phi\Uparrow"] & B \arrow[d, "id"] \\
B \arrow[r, "1"'] & B \arrow[r, "1"'] & B
\end{tikzcd}
$$

It follows that $\hat{d_1} \dashv i_p$ with identity counit.

A 2-monad of Street

Multiplication of monad $m : R^2 \Rightarrow R$ at (E, p) is given by the unique arrow $m_p : B/R(p) \to B/p$

$$
\begin{tikzcd}
B/R(p) \arrow[r, "\widehat{d_1^{\to}}"'] \arrow[r, dotted, bend left, "m_p"] \arrow[d, "\hat{\hat{p}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B/p \arrow[r, "\hat{d}_1"'] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
B^{\to\to} \arrow[r, "d_1^{\to}"'] \arrow[d, "d_0^{\to}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Phi\Uparrow"] & B \arrow[d, "1"] \\
B^{\to} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Phi\Uparrow"] & B \arrow[r, "1"'] \arrow[d, "1"] & B \\
B \arrow[r, "1"'] & B &
\end{tikzcd}
$$

with property that $R(p) \circ m_p = R^2(p)$ and $\hat{d}_1 \circ m_p = \hat{d}_1 \circ \widehat{d_1^{\to}}$, and moreover $\Phi_p \centerdot m_p = (\Phi_p \centerdot \widehat{d_1^{\to}}) \circ (\Phi \centerdot d_0^{\to} \centerdot \hat{\hat{p}})$, all inferred by universal property of comma object B/p.

A 2-monad of Street

Example

When $\mathcal{K} = \mathfrak{Cat}$, unit i_p takes an object N of E to the object $(N, 1_M)$ of B/p, where $M = p(N)$.

$$N \quad \to \quad \begin{tikzcd} & N \arrow[d, mapsto] \\ M \arrow[r, "1_M"'] & M \end{tikzcd}$$

Multiplication m_p takes an object $(N_2, f : M_0 \to M_1, g : M_1 \to M_2)$ of $B/R(p)$, where $M_2 = p(N_2)$, to the object $(N_2, f;g : M_0 \to M_2)$ of B/p.

$$\begin{tikzcd} & & N_2 \arrow[d, mapsto] \\ M_0 \arrow[r, "f"'] & M_1 \arrow[r, "g"'] & M_2 \end{tikzcd} \quad \to \quad \begin{tikzcd} & N_2 \arrow[d, mapsto] \\ M_0 \arrow[r, "f;g"'] & M_2 \end{tikzcd}$$

What does a pseudo-algebra of this monad look like?

Suppose $a : (B/p, R(p)) \to (E, p)$ is a pseudo-algebra for 2-monad R. It involves

1. 1-cell $a : B/p \to E$ such that $p \circ a = R(p)$
2. invertible 2-cell $\zeta_p : 1 \Rightarrow a \circ i_p$ such that $p \centerdot \zeta_p = id_p$.
3. invertible 2-cell $\theta_p : a \circ R(a) \Rightarrow a \circ m_p$ such that $p \centerdot \theta_p = id_{R^2(p)}$.

Additionally, ζ_p and θ_p satisfy coherence equations of pseudo-algebra a.

What does a pseudo-algebra of this monad look like?

Example

When $\mathcal{K} = \mathfrak{Cat}$,

1. 1-cell a gives us for any object $(N_1, f : M_0 \to M_1)$ of B/p an object $\mathrm{Pull}_f\, N_1$ of E over N_0.
2. 2-cell ζ_p gives us an isomorphism between N and $\mathrm{Pull}_{1_M}\, N$ over 1_N, whereby $M = p(N)$.
3. 2-cell θ_p gives us as isomorphism between $\mathrm{Pull}_f\, \mathrm{Pull}_g\, N_2$ and $\mathrm{Pull}_{f;g}\, N_2$ over 1_{M_0}.

   Additionally, following diagrams commute:

$$
\begin{tikzcd}
\mathrm{Pull}_f\, N_1 \arrow[d, "a_\bullet(R\zeta)"'] \arrow[dr, equal] & \\
\mathrm{Pull}_{1_{M_0}}\, \mathrm{Pull}_f\, N_1 \arrow[r, "\theta_\bullet Ri_p"'] & \mathrm{Pull}_f\, N_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Pull}_f\, \mathrm{Pull}_g\, \mathrm{Pull}_h\, N_3 \arrow[r] \arrow[d] & \mathrm{Pull}_{f;g}\, \mathrm{Pull}_h\, N_3 \arrow[d] \\
\mathrm{Pull}_f\, \mathrm{Pull}_{g;h}\, N_3 \arrow[r] & \mathrm{Pull}_{f;g;h}\, N_3
\end{tikzcd}
$$

Back to presentation

Unpacking them yields the following diagram in $\mathfrak{Top}$:

$$\begin{tikzcd}
& \underline{g}^*\mathcal{E} \arrow[rr, "\overline{g}"] \arrow[dl, "\overline{l(\alpha)}"'] \arrow[dd, "\underline{g}^*p"' near start] \arrow[rr, phantom, bend right=40, "\overline{\alpha}\Nearrow"] & & \mathcal{E} \arrow[dl, equal] \arrow[dd, "p"] \\
\underline{f}^*\mathcal{E} \arrow[rr, crossing over, "\overline{f}"' near start] \arrow[dd, "\underline{f}^*p"'] \arrow[dr, phantom, "l(\alpha)\Downarrow"] & & \mathcal{E} & \\
& \mathcal{A} \arrow[rr, "\underline{g}" near start] \arrow[dl, equal] \arrow[dr, phantom, "\underline{\alpha}\Nearrow"] & & \mathcal{S} \arrow[dl, equal] \\
\mathcal{A} \arrow[rr, "\underline{f}"'] & & \mathcal{S} \arrow[from=uu, crossing over, "p" near start] &
\end{tikzcd}$$

where obvious diagram of 2-cells commutes.

pack

Unpacking τ_0 yields the following diagram in $\mathfrak{Top}$:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\underline{f}^*\mathcal{E} \arrow[r, "\overline{I(id)}"] \arrow[r, bend right=50, "id"'] \arrow[r, phantom, shift right=3ex, "\Uparrow\overline{\tau_0}"] \arrow[rr, bend left=50, "\overline{f}"] \arrow[rr, phantom, shift left=6ex, "\Uparrow\overline{\alpha}"] \arrow[d, "\underline{f}^*p"'] & \underline{f}^*\mathcal{E} \arrow[r, "\overline{f}"] \arrow[d, "\underline{f}^*p"] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{A} \arrow[r, dotted, "id"] \arrow[r, bend right=50, "id"'] \arrow[r, phantom, shift right=3ex, "\Uparrow id"] & \mathcal{A} \arrow[r, "\underline{f}"] & \mathcal{S}
\end{tikzcd}
$$

pack

Unpacking $\tau_{\alpha,\beta}$ yields the following diagram in $\mathfrak{Top}$:

$$\begin{tikzcd}
\underline{h}^*\mathcal{E} \arrow[r, "\overline{I(\beta)}"] \arrow[rr, bend left, "\overline{I(\beta\circ\alpha)}"] \arrow[dr, "\cong"'] & \underline{g}^*\mathcal{E} \arrow[r, "\overline{I(\alpha)}"] \arrow[d, "\cong"] & \underline{f}^*\mathcal{E} \arrow[dl, "\cong"] \\
& \mathcal{A} &
\end{tikzcd}$$

Furthermore, we require

$$(\overline{\beta \circ \alpha}) \circ (\overline{f} \mathbin{\centerdot} \overline{\tau_{\alpha,\beta}}) = \overline{\beta} \circ (\overline{\alpha} \mathbin{\centerdot} I(\beta))$$
$$I(\beta \circ \alpha)\Downarrow \circ (f^* p \mathbin{\centerdot} \overline{\tau_{\alpha,\beta}}) = I(\beta)\Downarrow \circ (I(\alpha)\Downarrow \mathbin{\centerdot} \overline{I(\beta)})$$

pack

$\overline{I(\alpha \centerdot k)}$ is isomorphic to the bi-pullback of $\overline{I(\alpha)}$ along $\overline{k_f}$, which is to say the top left vertical square of the diagram commutes up to an isomorphism.

$$
\begin{tikzcd}
 & & (\underline{g}\,\underline{k})^*\mathcal{E} \arrow[dl, "\overline{I(\alpha \centerdot k)}"'] \arrow[dd, dotted, no head, "\cong"'] \arrow[drrrr, "\overline{gk}"] & & & & \\
 & (\underline{f}\,\underline{k})^*\mathcal{E} \arrow[dd, dotted, "\cong"'] \arrow[ddrrrr, "\overline{fk}"' near end] & & & & & \\
 & & \underline{k}^*\underline{g}^*\mathcal{E} \arrow[rr] \arrow[dl, "\underline{k}^*\overline{I(\alpha)}"'] \arrow[dd] & & \underline{g}^*\mathcal{E} \arrow[rr, "\overline{g}"] \arrow[dl] \arrow[dd] \arrow[dr, phantom, "\Nearrow \overline{\alpha}"] & & \mathcal{E} \arrow[dl, equal] \arrow[dd, "p"] \\
 & \underline{k}^*\underline{f}^*\mathcal{E} \arrow[rr, "\overline{k_f}"'] \arrow[dd, "\underline{k}^*\underline{f}^*p"'] & & \underline{f}^*\mathcal{E} \arrow[rr, "\overline{f}"'] \arrow[dd, "\underline{f}^*p"' near end] & & \mathcal{E} \arrow[dd, "p" near end] & \\
 & & \mathcal{S} \arrow[rr, "\underline{k}"] \arrow[dl, equal] & & \mathcal{A} \arrow[rr, "\underline{g}"] \arrow[dl, equal] \arrow[dr, phantom, "\Nearrow \underline{\alpha}"] & & \mathcal{S} \arrow[dl, equal] \\
 & \mathcal{B} \arrow[rr, "\underline{k}"'] & & \mathcal{A} \arrow[rr, "\underline{f}"'] & & \mathcal{S} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathcal{D} \arrow[d, "\overline{y}"] \arrow[ddl, bend right, "\overline{x}"'] & \\
 & \underline{g}^*\mathcal{E} \arrow[dl, dotted, "\overline{l(\alpha)}"'] \arrow[dr, "\overline{g}"] \arrow[dd] \arrow[dr, phantom, "\overline{\alpha}\,\Nearrow"', shift right=4] & \\
\underline{f}^*\mathcal{E} \arrow[rr, bend right, "\overline{f}"'] \arrow[dd, "\underline{f}^* p"'] \arrow[dr, phantom, "\cong"] & & \mathcal{E} \arrow[dd, "p"] \\
 & \mathcal{A} \arrow[dl, equal] \arrow[dr, "\underline{g}"] \arrow[dr, phantom, "\underline{\alpha}\,\Nearrow"', shift right=4] & \\
\mathcal{A} \arrow[rr, bend right, "\underline{f}"'] & & \mathcal{S}
\end{tikzcd}
$$

pack

With regards to models of a context $\mathbb{T}$, 2-category $\mathfrak{GTop}$ has a class of very special objects, namely a classifying topos $p : \mathcal{S}[\mathbb{T}] \to \mathcal{S}$, for each base topos $\mathcal{S}$, with the classifying property given by following equivalence of categories whereby $\mathcal{E}$ is an $\mathcal{S}$-topos:

$$\Phi : \mathfrak{BTop}/_{\mathcal{S}}\ (\mathcal{E}, \mathcal{S}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E}) : \Psi$$

which makes p the representable object for the index category $\underline{\mathbb{T}}$.