The Roots of Intuitionism and Constructivism in Mathematics

Part I

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January 2021

Outline

1. The Underlying POV
2. Philosophies of Mathematics
3. Foundations of Mathematics

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3. As Heidegger puts it in Being and Time we should ask again the question of "being".

Philosophies of Mathematics

What are philosophies of mathematics?

A philosophy of mathematics should explain three things:

- the ontology: what are the status of mathematical objects? where do they exist? etc.
- the epistemology: how can we access the knowledge of mathematical objects? what are the modes of the knowledge about them? etc.
- methodology: what are the ways to gain/produce mathematical knowledge? e.g. symbols, abstractions, generalization and extensions, etc.

The (Current) Threads in the Philosophy of Mathematics

The Philosophical Threads

- Structuralism
- Constructivism
- Formalism
- Logicism
- Fictionalism

Foundations of Mathematics

Foundations of mathematics?

- The search for securely grounding all of mathematics on a consistent theory started in the 19th century as a response to the “foundational crisis" in calculus (infinitesimals, etc.) and geometry (non-Euclidean geometries).
- In the early 20th century, Zermelo, and Fraenkel had axiomatized Cantor’s theory of sets.
- Fun fact: ZFC is not proven to be consistent.
- Nevertheless, until today ZFC has been considered as the dominant foundation in which almost all of mathematics can be encoded. Most philosophers of mathematics believe in the superiority of ZFC over other foundations.

Figure: Foundation as the Cosmos of Indian Myth

The Axioms of ZFC Set Theory

1. $\forall a_1 \forall a_2 \; (\forall b \, (b \in a_1 \Leftrightarrow b \in a_2) \Rightarrow a_1 = a_2)$
2. $\forall a_1 \forall a_2 \; \exists c \, \forall b \, \big(b \in c \Leftrightarrow (b = a_1 \;\vee\; b = a_2)\big)$
3. $\forall a \; \exists d \, \forall c \, \big(c \in d \Leftrightarrow \exists b \, (b \in a \;\wedge\; c \in b)\big)$
4. $\forall a \; \exists d \, \forall b \, \big(b \in d \Leftrightarrow \forall c \, (c \in b \Rightarrow c \in a)\big)$
5. $\forall a \; \exists c \, \forall b \, \big(b \in c \Leftrightarrow b \in a \;\wedge\; \Phi[b]\big)$
6. $\exists a \colon \big(\varnothing \in a \;\wedge\; \forall b \, (b \in a \Rightarrow b \cup \{b\} \in a)\big)$
7. $\forall x \; \exists! y \; \phi[x, y] \Rightarrow \forall a \; \exists d \, \forall c \, \big(c \in d \Leftrightarrow \exists b \, (b \in a \;\wedge\; \phi[b, c])\big)$
8. $\forall a \; \Big(a \neq \varnothing \;\wedge\; \forall b \, (b \in a \Rightarrow b \neq \varnothing) \;\wedge\; \forall b_1 \forall b_2 \, (b_1 \neq b_2 \;\wedge\; \{b_1, b_2\} \subseteq a \Rightarrow b_1 \cap b_2 = \varnothing)$
   $\Rightarrow \exists d \forall b \, \big(b \in a \to \exists c \, (b \cap d = \{c\})\big)\Big)$
9. $\forall a \; \Big(a \neq \varnothing \Rightarrow \exists b \, \big(b \in a \;\wedge\; \forall c \, (c \in b \Rightarrow c \notin a)\big)\Big)$

Figure: The Axioms of ZFC Set Theory

All mathematics is derivable from these axioms!

Remarkable fact: "In principle" all mathematics can be derived from these axioms using the derivation rule of the first order logic. This is in virtue of more than 100 years rule of formalist practice in mathematics.

$$\dfrac{\dfrac{((\phi \wedge \psi) \vdash_{\vec{x},y} \phi)}{((\exists y)(\phi \wedge \psi) \vdash_{\vec{x}} \phi)} \qquad \dfrac{\dfrac{((\phi \wedge \psi) \vdash_{\vec{x},y} \psi) \quad \dfrac{((\exists y)\psi \vdash_{\vec{x}} (\exists y)\psi)}{(\psi \vdash_{\vec{x},y} (\exists y)\psi)}}{((\phi \wedge \psi) \vdash_{\vec{x},y} (\exists y)\psi)}}{((\exists y)(\phi \wedge \psi) \vdash_{\vec{x}} (\exists y)\psi)}}{((\exists y)(\phi \wedge \psi) \vdash_{\vec{x}} (\phi \wedge (\exists y)\psi))}$$

Figure: The proof tree of the Frobenius axiom

Alternative Foundations

- Dependent Type Theory
- Homotopy Type Theory (HoTT) / Univalent Foundation (UF)
- Category Theory
- etc.

The role/use of foundation

What do we expect from a foundational system?

- Proving consistency?
- Organization of different branches of mathematics?
  (HoTT library module dependency graph) compare to
  (Wolfram's empirical metamathematics of Euclid and beyond)
- Hierarchical organization of successive abstractions (e.g. category theory)
- As a grammar for a language?
- A formalization tool?
- A practical aid?
- A pedagogical aid?
- An immune system for sharp mathematical reasoning?

Anti-foundationalism and Foundational Pluralism

People interested in categorical foundations are interested in formalizing mathematics in a way that fits how mathematicians actually think, and different mathematicians think in different ways at different times. That is why we tend to prefer what is called a "multi-foundational" approach. Personally I don't think the metaphor of "foundations" is even appropriate for this approach. I prefer a word like "entrance". A building has one foundation, which holds up everything else. But mathematics doesn't need anything to hold it up: there is no "gravity" that pulls mathematics down and makes it collapse. What mathematics needs is "entrances": ways to get in. And it would be very inconvenient to have just one entrance.
(John Baez - on Foundations of Mathematics, n-Category Cafe, 2016)

A New Proposal

Mike Shulman proposes a kind of foundational pluralism in the sense of the following analogy:

- "Foundations" similar to Turing-complete programming languages: a given Turing-complete PL can be interpreted to any other Turing-complete PL and compiled to many different architectures. However some languages are superior to others for certain kinds of purposes.
- In this way one can think of ZFC set theory as the machine language (unreadable to humans), and foundations such category theory as higher languages (such as C, Java, Haskell, etc.). Of course, it is quite useful to keep ZFC in the background since after all the code in the higher level languages is ultimately be compiled into the machine language for your computer to do anything at all.