The comprehension construction and fibrations of toposes

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Introduction

- Part A: motivating categorical fibrations from logic
- Part B: fibrations of toposes

Part A: categorical fibrations from logic

- Recall that the Axiom Schema of Comprehension of ZF:

$$\forall w_1, \ldots, w_n \, \forall A \, \exists B \, \forall x \, (x \in B \Leftrightarrow [x \in A \wedge \varphi(x, w_1, \ldots, w_n, A)])$$

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- We get a predicate $P\colon A \to \{\text{True}, \text{False}\}$ once extra parameters $w_1, ..., w_n$ have been specified as sets.
- Comprehension essentially says that given a set A and a predicate P with a free variable x whose values range over A, we can find a subset B of A whose members are precisely the members of A that satisfy P.
- By the axiom of extensionality the set B is unique and is denoted by $\{a \in A : P(a) = \text{True}\}$.

A categorical comprehension scheme

- Categorifying "comprehension" naturally leads to the notion of subobject classifier (of an elementary topos, say $\mathscr{E}$).

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- Categorifying "comprehension" naturally leads to the notion of subobject classifier (of an elementary topos, say $\mathscr{E}$).
- True is represented as a monomorphism $\text{True}\colon 1 \rightarrowtail \Omega$, and $\{a \in A : P(a) = \text{True}\}$ is given by the object B of the pullback of $\text{True}\colon 1 \rightarrowtail \Omega$ and $P\colon A \to \Omega$.

$$
\begin{tikzcd}
B \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\text{True}"] \\
A \arrow[r, "P"'] & \Omega
\end{tikzcd}
$$

- (Lawvere, 1970) For a *hyperdoctrine* $\mathcal{P}$, a comprehension schema is a pair of adjoint functors

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathbb{T}/X \arrow[l, bend right, "L"'] \end{tikzcd}$$

- It is often the case that a comprehension scheme for a hyperdoctrine arises in situations when we have
    1. A factorization system $(\mathcal{E}, \mathcal{M})$ on the category $\mathbb{T}$.
    2. An object Ω in $\mathbb{T}$ which classifies (equivalence classes of) $\mathcal{M}$-objects, i.e. we have natural isomorphisms

$$\mathbb{T}(X, \Omega) \cong \mathcal{M}(X)$$

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- $\mathbb{T}$: elementary topos
  $\mathcal{P} := \mathbb{T}(-, \Omega_{\mathbb{T}})$, and $(\mathcal{E}, \mathcal{M})$: the usual epi-mono factorization.
  Therefore, $\mathcal{P}(X) \cong \mathbb{T}(X, \Omega) \cong \mathcal{M}(X)$. The unit (η) of adjunction above is given by the image factorization:

$$\begin{tikzcd} Y \arrow[d, two heads, "e"'] \arrow[dr, "f"] & \\ Y_f \arrow[r, tail, "m"'] & X \end{tikzcd} \qquad \text{with} \quad \eta(f) = e.$$

n-Categorical levels

- Consider the following chain of categories (equipped with their notion of structural identity of objects) and adjoint functors

$$
\begin{tikzcd}
(\{*\},=) \arrow[r, bend left=60, "\emptyset"] \arrow[r, bend right=60, "\{\emptyset\}"'] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"] & (\{\emptyset \to \{\emptyset\}\},=) \arrow[l, "||-||_{-2}"'] \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"] & (\mathcal{S}\mathsf{et},\cong) \arrow[l, "||-||_{-1}"']
\end{tikzcd}
$$

n-Categorical levels

- The above chain of categories can be extended to the following chain of 2-categories and (strict) 2-functors.

$$
\begin{tikzcd}[column sep=huge]
1 \arrow[r, bend left=70, "\bot"] \arrow[r, bend right=70, "\top"'] \arrow[r, phantom, bend left=35, "\perp"] \arrow[r, phantom, bend right=35, "\perp"] & \{\bot \to \top\} \arrow[l, "\|-\|_{-2}" description] \arrow[r, bend left=70] \arrow[r, bend right=70] \arrow[r, phantom, bend left=35, "\perp"] \arrow[r, phantom, bend right=35, "\perp"] & \mathcal{S}\mathrm{et} \arrow[l, "\|-\|_{-1}" description] \arrow[r, bend left=70, "(-)^{\mathrm{d}}"] \arrow[r, bend right=70, "(-)^{\mathrm{ind}}"'] \arrow[r, phantom, bend left=35, "\perp"] \arrow[r, phantom, bend right=35, "\perp"] & (\mathfrak{Cat}, \simeq) \arrow[l, "\|-\|_{0}" description]
\end{tikzcd}
$$

n-Categorical levels

- And the chaing of 3-categories and 3-functors:

$$
\begin{tikzcd}[column sep=huge]
1 \arrow[r, bend left=60, "\perp"] \arrow[r, bend right=60, "\top"'] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"]
& \{\perp \to \top\} \arrow[l, "\|-\|_{-2}"description] \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"]
& \mathcal{S}\mathrm{et} \arrow[l, "\|-\|_{-1}"description] \arrow[r, bend left=60, "(-)^{\mathrm{d}}"] \arrow[r, bend right=60, "(-)^{\mathrm{ind}}"'] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"]
& (\mathfrak{Cat}, \simeq) \arrow[l, "\|-\|_{0}"description] \arrow[r, bend left=60, "(-)^{\mathrm{d}}"] \arrow[r, bend right=60, "(-)^{\mathrm{ind}}"'] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"]
& (\mathbf{2}\,\mathfrak{Cat}_{psd}, \simeq) \arrow[l, "\|-\|_{1}"description]
\end{tikzcd}
$$

A 2-categorical comprehension scheme

- Note that in the categorical comprehension we relied on the structure of subobject classifier $\mathrm{True} \colon 1 \rightarrowtail \Omega$ whose *fibres* are *subsingletons*, i.e. they are (-1)-level "types".

A 2-categorical comprehension scheme

- Note that in the categorical comprehension we relied on the structure of subobject classifier $\mathrm{True}\colon 1 \rightarrowtail \Omega$ whose *fibres* are *subsingletons*, i.e. they are (-1)-level "types".
- (Street and Walters, 1978) and (Weber, 2007) observed that for a 2-categorical comprehension the generic subobject $\mathrm{True}\colon 1 \rightarrowtail \Omega$ should be replaced with the discrete opfibration $U\colon\ \mathcal{S}\mathrm{et}_\bullet \to \mathcal{S}\mathrm{et}$ whose *fibres* are *discrete*, aka sets, aka 0-level "types".

A 2-categorical comprehension scheme

- For a (small) category $\mathcal{C}$ and a functor $\mathbb{X} \colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, we have

$$
\begin{tikzcd}
 & & \mathcal{C} \rtimes \mathbb{X} \arrow[r, "\pi_1"] \arrow[d, dashed, "\pi_{\mathbb{X}}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{S}\mathrm{et}_\bullet \arrow[d, "U"] & & \mathcal{C} \rtimes \mathbb{X} \arrow[d, dashed, "\pi_{\mathbb{X}}"'] \\
\mathcal{C} \arrow[r, "\mathbb{X}"'] & \mathcal{S}\mathrm{et} \arrow[r, phantom, "\overset{\text{Pullback along U}}{\rightsquigarrow}"] & \mathcal{C} \arrow[r, "\mathbb{X}"'] & \mathcal{S}\mathrm{et} \arrow[r, phantom, "\overset{\text{forgetful}}{\rightsquigarrow}"] & & \mathcal{C}
\end{tikzcd}
$$

The composite process is the well-known *Grothendieck construction*.

A 2-categorical comprehension scheme

- An equivalence of categories:

$$\begin{tikzcd} \mathcal{F} \arrow[d, "p"'] \\ \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd}[column sep=huge] {} \arrow[r, "\text{Functor of fibres}"] & {} \\ {} & {} \arrow[l, "\text{Grothendieck construction}"] \end{tikzcd} \qquad \mathcal{C} \xrightarrow{\mathbb{P}} \mathcal{S}\mathrm{et}$$

discrete opfibrations

functors

A 2-categorical comprehension scheme

- $U\colon \mathcal{S}\mathrm{et}_\bullet \to \mathcal{S}\mathrm{et}$ classifies discrete opfibrations, and for a discrete fibration $p\colon \mathcal{E} \to \mathcal{B}$ we have a functor

  $$\mathrm{cmp}_{p,\mathcal{C}}\colon \mathfrak{Cat}(\mathcal{C}, \mathcal{B}) \xrightarrow{\simeq} \mathrm{do}\mathcal{F}\mathrm{ib}\,/\mathcal{C}$$

  which is an equivalence of categories and natural in $\mathcal{C}$.

  Note: A natural transformation $\alpha\colon \mathbb{X} \Rightarrow \mathbb{Y}\colon \mathcal{C} \rightrightarrows \mathcal{B}$ is taken to a functor $\ell(\alpha)\colon \mathbb{X}^*p \to \mathbb{Y}^*p$ in $\mathrm{do}\mathcal{F}\mathrm{ib}\,/\mathcal{C}$.

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- The equivalence

  $$\mathfrak{Cat}(\mathcal{C}, \mathcal{S}\mathsf{et}) \xrightarrow{\sim} \mathsf{do}\mathcal{F}\mathsf{ib}\,/\mathcal{C}$$

  is the counterpart of the natural isomorphism

  $$\mathbb{T}(X, \Omega) \cong \mathcal{P}(X)$$

  .

Factorization systems of discrete categorical comprehension

(Street and Walters, 1978) give two factorization systems corresponding to the discrete categorical comprehension construction:

1. (initial functors, discrete opfibrations)
2. (final functors, discrete fibrations)

They should be seen as generalizations of (epi, mono) factorization system.

- The discrete comprehension construction above can be imitated in any 2-category: one would need to require that discrete fibrations to be carrable, i.e. their strict pullback along any 1-morphisms exists.

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- We can generalized the comprehension construction from 0-type valued predicates (i.e. functors to $\mathcal{S}\mathrm{et}$) to 1-type valued predicates (i.e. functors to $\mathfrak{Cat}$). To formulate the comprehension construction for 1-type valued predicates we need the notion of Grothendeick opfibration: here the fibres are (in general non-discrete) categories which covariantly depend on the objects of the base.

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- More generally, we can perform the comprehension construction for the 1-type valued predicates along *Johnstone fibrations* internal to *any* 2-category.

Johnstone's fibrations in 2-categories

Definition (P. Johnstone, 93)

Suppose $\mathfrak{K}$ is a 2-category. A 1-morphism $p\colon E \to B$ is an (internal) **opfibration** in $\mathfrak{K}$ if

1. it is bicarrable, and
2. for any 2-morphism $\alpha\colon f \Rightarrow g\colon A \rightrightarrows B$ in $\mathfrak{K}$, there exist a 1-morphism $\ell(\alpha)\colon f^*E \to g^*E$, a 2-morphism $\widetilde{\alpha}\colon p^*f \Rightarrow p^*g \circ \ell(\alpha)$, and an invertible 2-morphism $\tau(\alpha)\colon f^*p \Rightarrow g^*p \circ \ell(\alpha)$ satisfying *five coherence axioms*.

$$
\begin{tikzcd}
& g^*E \arrow[rr, "p^*g"] \arrow[dd, dashed, "g^*p" near start] & & E \arrow[dd, "p"] \\
f^*E \arrow[rr, "p^*f" near end] \arrow[dd, "f^*p"'] & & E \arrow[ur, equal] \arrow[dd, "p" near start] & \\
& A \arrow[rr, "g"] & & B \\
A \arrow[ur, equal] \arrow[rr, "f"'] \arrow[urr, phantom, "\alpha \Nearrow"] & & B \arrow[ur, equal] &
\end{tikzcd}
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$$\begin{tikzcd}
 & g^*E \arrow[rr, "p^*g"] \arrow[dd, dashed, "g^*p"] & & E \arrow[dl, equal] \arrow[dd, "p"] \\
f^*E \arrow[rr, "p^*f"] \arrow[dd, "f^*p"'] & & E \arrow[dd, "p"] & \\
 & A \arrow[rr, "g"] \arrow[dl, equal] \arrow[dr, phantom, "\alpha \Nearrow"] & & B \arrow[dl, equal] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & g^*E \arrow[rr, "p^*g"] \arrow[dd, dashed, "g^*p"] \arrow[dr, phantom, "\tilde{\alpha} \Nearrow"] & & E \arrow[dl, equal] \arrow[dd, "p"] \\
f^*E \arrow[ur, "\ell(\alpha)"] \arrow[rr, "p^*f"] \arrow[dd, "f^*p"'] \arrow[dr, phantom, "\cong_{\tau(\alpha)}"] & & E \arrow[dd, "p"] & \\
 & A \arrow[rr, "g"] \arrow[dl, equal] \arrow[dr, phantom, "\alpha \Nearrow"] & & B \arrow[dl, equal] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

Johnstone's fibrations in 2-categories

- The five coherence conditions are about the coherence of the lifting structure with respect to the unit and composition of 2-morphisms targeted at B, the whiskering of 2-morphisms targeted at B with compatible 1-morphisms, and finally the cartesianness of the lift $\widetilde{\alpha}$.

Johnstone's fibrations in 2-categories

- The five coherence conditions are about the coherence of the lifting structure with respect to the unit and composition of 2-morphisms targeted at B, the whiskering of 2-morphisms targeted at B with compatible 1-morphisms, and finally the cartesianness of the lift $\widetilde{\alpha}$.
- (Hazratpour and Vickers, 2018) They can be summarized as a single condition in terms of 1-cartesianness and 2-cartesianness with respect to the 2-functor $\mathbf{cod}\colon \mathfrak{K}_{\mathcal{D}} \to \mathfrak{K}$.

$$
\begin{tikzcd}
\mathfrak{K}_{\mathcal{D}} \arrow[rr, hook] \arrow[dr, "\mathbf{cod}"'] & & \mathbf{cyl}_{\cong}(\mathfrak{K}) \arrow[dl, "\mathbf{cod}"] \\
& \mathfrak{K} &
\end{tikzcd}
$$

where $\mathcal{D}$ is a distinguished class of "display morphisms" in $\mathfrak{K}$ such that

1. Every identity 1-morphism is in $\mathcal{D}$.
2. If $x\colon \overline{x} \to \underline{x}$ is in $\mathcal{D}$, and $f\colon \underline{y} \to \underline{x}$ in $\mathfrak{K}$, then there is some bipullback y of x along $\underline{f}$ such that $y \in D$.

- From the structure of Johnstone fibration we get a pseudo-functor

$$\mathrm{cmp}_{p,A} \colon \mathfrak{K}(A, B) \to \mathsf{op}\mathfrak{F}\mathsf{ib}(\mathfrak{K}) \mathbin{/\!\!/} A$$

(Note: $\mathsf{op}\mathfrak{F}\mathsf{ib}(\mathfrak{K}) \mathbin{/\!\!/} A$ is the pseudo-slice (aka homotopy slice) 2-category.)

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- Note that the pseudo functor $\mathrm{cmp}_{p,A}$ is not an equivalence. (e.g. a discretely equifibred category over $\mathbf{2}$.)
- p is called a *univalent opfibration* if the equivalence of fibres induces an iso 2-morphism in $\mathfrak{K}$.

$$\begin{tikzcd}
\mathfrak{Core}(\mathfrak{K}(A, B)) \arrow[r] \arrow[d, tail] & \mathfrak{Core}(\mathsf{op}\mathfrak{F}\mathsf{ib}(\mathfrak{K}) \mathbin{/\!\!/} A) \arrow[d, tail] \arrow[l, bend right, dashed] \\
\mathfrak{K}(A, B) \arrow[r] & \mathsf{op}\mathfrak{F}\mathsf{ib}(\mathfrak{K}) \mathbin{/\!\!/} A
\end{tikzcd}$$

Fibrations of toposes

- Crucially, Johnstone's definition does not require strictness of the 2-category nor the existence of the structure of strict pullbacks and comma objects.

- This definition is most suitable for 2-categories with weak limits such as the *2-category of toposes* where the limits are weak and diagrams are in general not expected to commute strictly.

Classifying toposes as representing objects

- Let $\mathbb{T}$ be a geometric theory, and $\mathcal{S}$ an elementary topos with nno. Consider the pseudo-functor

$$\mathbb{T}\text{-}\mathbf{Mod}\text{-} : (\mathcal{B}\mathfrak{Top}/\mathcal{S})^{\mathrm{op}} \to \mathfrak{Cat}$$

- To an $\mathcal{S}$-topos $\mathscr{E}$ it assigns the category $\mathbb{T}\text{-}\mathbf{Mod}\text{-}\mathscr{E}$ of models $\mathbb{T}$ in $\mathscr{E}$.
- To a geometric morphism $(f^*, f_*) \colon \mathscr{F} \to \mathscr{E}$ of $\mathcal{S}$-toposes it assigns the functor $f^* \colon \mathbb{T}\text{-}\mathbf{Mod}\text{-}\mathscr{E} \to \mathbb{T}\text{-}\mathbf{Mod}\text{-}\mathscr{F}$.
- Note that $\mathbb{T}\text{-}\mathbf{Mod}\text{-}(f \circ g) \cong (\mathbb{T}\text{-}\mathbf{Mod}\text{-}f) \circ (\mathbb{T}\text{-}\mathbf{Mod}\text{-}g)$
- The classifying topos $\mathcal{S}[\mathbb{T}]$ of a geometric theory/context $\mathbb{T}$ can be seen as a representing object for this pseudo-functor, i.e.

$$\mathcal{B}\mathfrak{Top}/\mathcal{S}(\mathscr{E}, \mathcal{S}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathbf{Mod}\text{-}\mathscr{E}$$

naturally in $\mathscr{E}$.

Given
$p\colon \mathscr{E} \to \mathcal{S}$: a bounded geometric morphism,
$U\colon \mathbb{T}_1 \to \mathbb{T}_0$: an extension of geometric theories,
and M is a (strict) model of theory $\mathbb{T}_0$ in the base topos $\mathcal{S}$,
then define

$$\underline{\mathbb{T}_1/M}(\mathscr{E})\colon = \text{(strict) models of } \mathbb{T}_1 \text{ in } \mathscr{E} \text{ which reduce to } p^\star M \text{ via } U$$

Fibrations of toposes from fibrational extension of theories

Theorem (S.H, S. Vickers, 2018)

If $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ *is a (op)fibrational extension of theories, and* M *is any model of* $\mathbb{T}_0$ *in an elementary topos* $\mathbb{S}$, *then* $p\colon \mathbb{S}[\mathbb{T}_1/M] \to \mathbb{S}$ *is an (op)fibration of toposes.*

Fibrations of toposes from fibrational extension of theories

Theorem (S.H, S. Vickers, 2018)

If $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ is a (op)fibrational extension of theories, and M is any model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, then $p \colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is an (op)fibration of toposes.

Moreover, for any geometric (*not necessarily bounded*) morphism $\underline{f} \colon \mathcal{A} \to \mathcal{S}$, the classifying topos $\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M]$ is got by bipullback of p along $\underline{f}$:

$$\begin{tikzcd}
\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M] \arrow[r] \arrow[d, "\underline{f}^* p"'] \arrow[dr, phantom, "\cong"] & \mathcal{S}[\mathbb{T}_1/M] \arrow[d, "p"] \\
\mathcal{A} \arrow[r, "\underline{f}"'] & \mathcal{S}
\end{tikzcd}$$

Sina Hazratpour and Steven Vickers. "Fibrations of AU-contexts beget fibrations of toposes". In: *arXiv:1808.08291* (2018). Submitted to Theory and Application of Categories (TAC).

We actually proved the theorem above for the more general case of “AU-contexts” which are the logical theories of Arithmetic Universes.

Arithmetic Universe(AU)= **list-arithmetic pretopos**: a category with finite limits, stable disjoint coproducts, stable effective quotients by monic equivalence relations and parameterized list-objects.

Maria Emilia Maietti. "Joyal's arithmetic universe as list-arithmetic pretopos". In: *Theory and Applications of Categories* 24 (2010), pp. 39–83

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The above definition of AU parallels Giraud's characterization of relative Grothendieck toposes, except that AUs have only finitary fragment of geometric logic, and instead of infinitary disjunctions being supplied extrinsically by a base topos (e.g. the structure of small-indexed coproducts), we have sort constructors for **parametrized list object** that allow some, infinities intrinsically: e.g. point-free continuum.

	Arithmetic Universes	Grothendieck toposes
Classifying category	$\mathbf{AU}\langle\mathbb{T}\rangle$	$\mathscr{S}[\mathbb{T}]$
$\mathbb{T}_1 \to \mathbb{T}_2$	$\mathbf{AU}\langle\mathbb{T}_2\rangle \to \mathbf{AU}\langle\mathbb{T}_1\rangle$	$\mathscr{S}[\mathbb{T}_1] \to \mathscr{S}[\mathbb{T}_2]$
Base	Base independent	Base $\mathscr{S}$
Infinities	Intrinsic; provided by *List* e.g. $N = List(1)$	Extrinsic; got from $\mathscr{S}$ e.g. infinite coproducts
Results	A single result in AUs	A family of results by varying $\mathscr{S}$

In our approach geometric theories are replaced by AU-contexts – presented by finite-limit-colimit sketches with a parametrized list object – and geometric morphisms are replaced by AU-functors, corresponding to the inverse image functors.

In our approach geometric theories are replaced by AU-contexts – presented by finite-limit-colimit sketches with a parametrized list object – and geometric morphisms are replaced by AU-functors, corresponding to the inverse image functors.

AU-contexts provide a base-independent model for generalized point-free spaces in the sense that they form a 2-category $\mathfrak{Con}$ which *gets embedded into* $\mathcal{B}\mathfrak{Top}/\mathcal{S}$, for all base toposes $\mathcal{S}$, via their classifying AUs.

Our AU technique has the many advantages to topos techniques of Johnstone: Our proofs are

- conceptually stronger,
- predicative,
- base-independent, and
- finitary and decidable, and therefore susceptible to formalization.

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Basic examples

- Etale morphisms of toposes are opfibrations.

  $\mathcal{S}_0$: an elementary topos,
  $\mathscr{S}$: a bounded $\mathcal{S}_0$ topos
  $\mathbb{T}_0 = \mathbb{O} = \{X\}$ and $\mathbb{T}_1 = \mathbb{O}_\bullet = \{X, x : X\}$

$$\begin{tikzcd}
\mathscr{S}/M \simeq \mathscr{S}[\mathbb{T}_1/M] \arrow[r] \arrow[d, "M^* p"'] & \mathcal{S}_0[\mathbb{O}_\bullet] = \mathcal{S}_0[X][\mathbb{T}_1/X] \arrow[d, "p"] \\
\mathscr{S} \arrow[r, "M"'] & \mathcal{S}_0[\mathbb{O}]
\end{tikzcd}$$

Basic examples

- Fibrewise Stone bundles (spectra of Boolean Algebras) are fibrations constructed from the extension $\mathbb{BA}_\bullet$ of the theory $\mathbb{BA}$ of Boolean Algebras.
- Internal Algebraic dcpos as opfibrations
- Spectral spaces as fibrations
- SFP domains as bifibrations
- Internal groups equipped with an action as fibrations
- Internal categories equipped with a torsor as opfibrations
- Internal modules as bifibrations
- Bag domains as opfibrations
- ...

Some further questions in this direction

- What is the class of topos (op)fibrations induced from fibrational extension of theories?
- Conjecture: fibrations of toposes and local geometric morphisms should form a weak bicategorical factorization system for toposes.
- To what extent AUs can replace Grothendieck toposes as models of spaces?
- (Riehl and Verity, 2017) provide a comprehension construction for objects of ∞-cosmoi which generalizes Lurie's straigthening and unstraigthening construction of quasi-categories (Yoneda Lemma for quasi-categories). What is the higher topos formulation of their comprehension construction and how does it relate to fibration of higher toposes?

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End!

Thanks for your attention!