Towards the Groupoid Model of HoTT in Lean 4

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joint work with Steve Awodey and Mario Carneiro

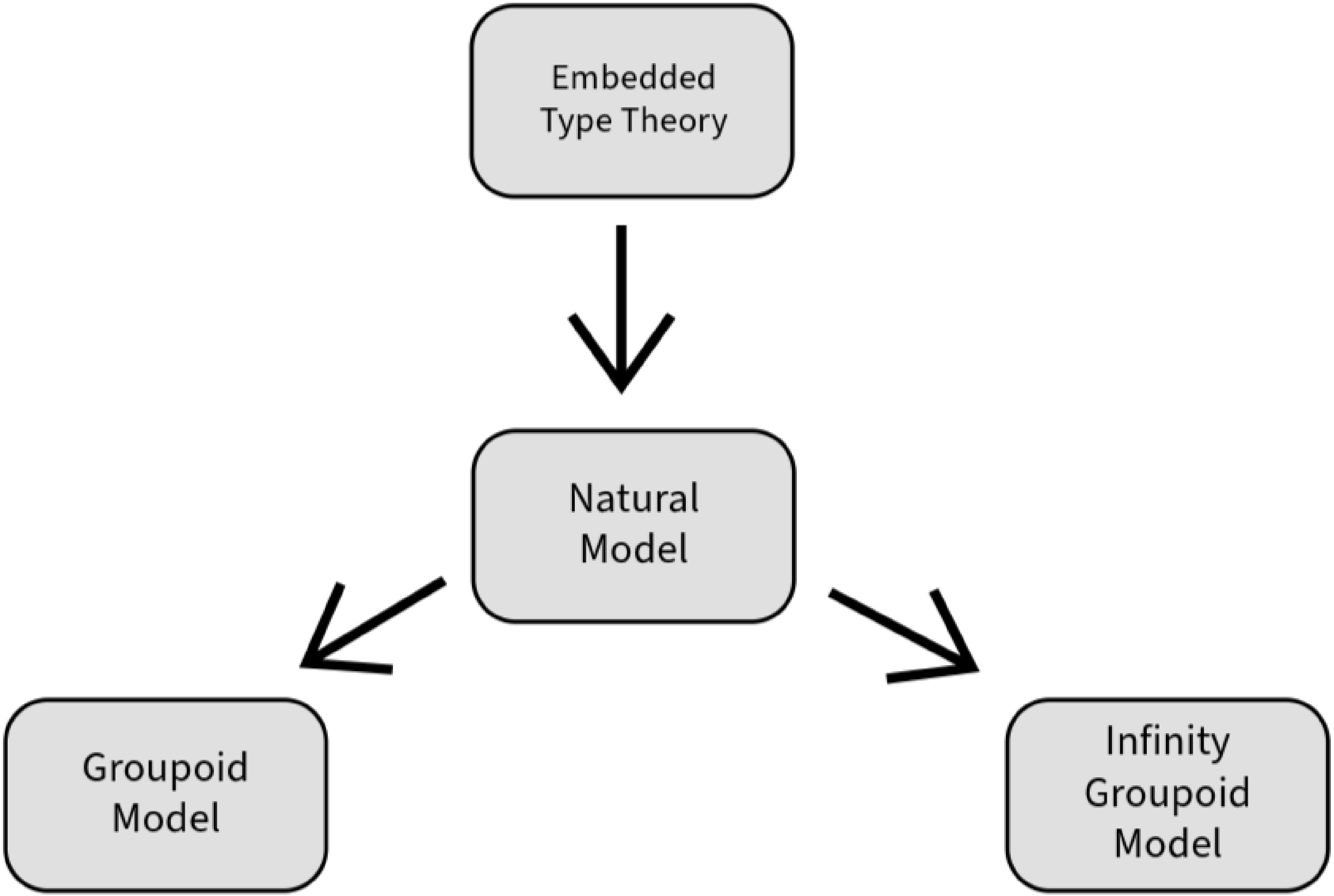
Previous work in bringing HoTT to Lean

special support for Homotopy Type Theory in Lean 2

- `github.com/leanprover/lean2/blob/master/hott/hott.md`
- Serre spectral sequence in Lean 2, by Floris van Doorn
  `github.com/cmu-phil/Spectral`

HoTT in Lean 3 by Gabriel Ebner, et al: github.com/gebner/hott3

- port of the Lean 2 HoTT library to Lean 3.
- The Lean 3 kernel is inconsistent with univalence.
- No modifications to the Lean kernel.


Embedded
Type Theory
Natural
Model
Groupoid
Model
Infinity
Groupoid
Model

Outline

i. Polynomial Functors : A prelude

ii. Natural models of HoTT

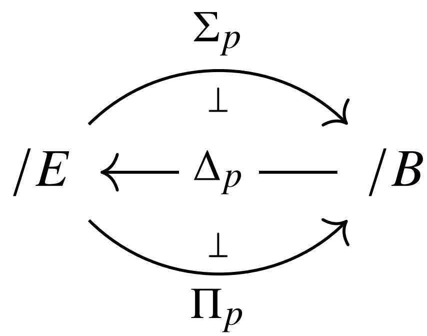
iii. The groupoid model of HoTT

iv. HoTT_0: The embedded type theory

Polynomial Functors : A prelude

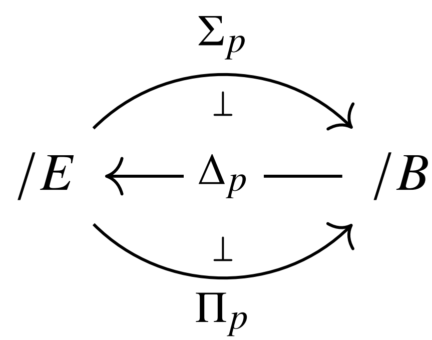
Let $\mathbb{C}$ be a category with finite limits.

Let $p : E \to B$ be an exponentiable morphism in $\mathbb{C}$. Thus:



When $B = 1$ is the terminal object, we write

$$\Sigma_E \vdash \Delta_E \vdash \Pi_E$$




```
@[inherit_doc]
prefix:90 "Σ_ " => Over.forget

@[inherit_doc]
prefix:90 "Δ_ " => Over.pullback

@[inherit_doc]
prefix:90 "Π_ " => CartesianExponentiable.functor
```

The polynomial endofunctor $P_p : \mathbb{C} \to \mathbb{C}$ associated to p is the composite

$$
\begin{array}{ccccc}
 & \mathbb{C}/E & \xrightarrow{\Pi_p} & \mathbb{C}/B & \\
 {\scriptstyle \Delta_E} \nearrow & & & & \searrow {\scriptstyle \Sigma_B} \\
\mathbb{C} & & \xrightarrow[P_p]{\qquad\qquad\qquad\qquad} & & \mathbb{C}
\end{array}
$$

In the internal language of $\mathbb{C}$,

$$P_p(X) \;=\; \sum_{b:B} X^{E(b)}$$

```
def functor [HasBinaryProducts C] (P : UvPoly E B) :
    C ⥤ C :=
  (Δ_ E) ⋙ (Π_ P.p) ⋙ (Σ_ B)
```

Natural models of HoTT

```
variable {Ctx : Type u} [SmallCategory Ctx] [HasTerminal
Ctx]

notation:max "y(" Γ ")" => yoneda.obj Γ

variable (Ctx) in

class NaturalModelBase where
Tm : Psh Ctx
Ty : Psh Ctx
tp : Tm ⟶ Ty
ext (Γ : Ctx) (A : y(Γ) ⟶ Ty) : Ctx
disp (Γ : Ctx) (A : y(Γ) ⟶ Ty) : ext Γ A ⟶ Γ
var (Γ : Ctx) (A : y(Γ) ⟶ Ty) : y(ext Γ A) ⟶ Tm
disp_pullback {Γ : Ctx} (A : y(Γ) ⟶ Ty) :
   IsPullback (var Γ A) (yoneda.map (disp Γ A)) tp A
```

Let $tp : Tm \to Ty$ be the representable typing natural transformation of a natural model.

Consider the associated polynomial endofunctor $P_{tp} : Psh(\mathbb{C}\mathbb{t}\mathbb{x}) \to Psh(\mathbb{C}\mathbb{t}\mathbb{x})$ defined as

$$\Sigma_{Ty} \circ \Pi_{tp} \circ \Delta_{Tm}.$$

Thus, internally,

$$P_{tp}(X) \ = \ \sum_{A:Ty} X^{[A]}$$

Applying P to Ty itself gives the object of type families:

$$P_{tp}(Ty) \ = \ \sum_{A:Ty} Ty^{[A]}$$

Theorem (Awodey, 2017):

The natural model models the rules of dependent type theory for the type formers Σ, Π and the universe.

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Theorem (Garner): The map $tp : Tm \to Ty$ models the rules for intensional identity types just if there are maps (i, Id) making the following diagram commute

$$\begin{array}{ccc} Tm & \xrightarrow{I} & Tm \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle tp} \\ Tm \times_{Ty} Tm & \xrightarrow[Id]{} & Ty \end{array}$$

and the induced comparison square a uniform weak pullback.

```
class NaturalModelPi where
  Pi : (P tp).obj Ty ⟶ M.Ty
  lam : (P tp).obj Tm ⟶ M.Tm
  Pi_pullback : IsPullback lam ((P tp).map tp) tp Pi
```

```
class NaturalModelSigma where
  Sig : (P tp).obj Ty ⟶ M.Ty
  pair : (P tp).obj Tm ⟶ M.Tm
  Sig_pullback : IsPullback pair ((uvPoly tp).comp
(uvPoly tp)).p tp Sig
```

```
variable [M : NaturalModelBase Ctx]

class NaturalModelIdBase where
  Id : pullback tp tp ⟶ M.Ty
  i : Tm ⟶ M.Tm
  Id_commute : δ ≫ Id = i ≫ tp
```

```
irreducible_def NaturalModelIdData :=
{ J : pb2 ⟶ (P q).obj M.Tm // J ≫ ε = 1 _ }

class NaturalModelId extends NaturalModelIdBase Ctx where
data : NaturalModelIdData Ctx

def NaturalModelId.J [NaturalModelId Ctx] :
   pb2 ⟶ (P q).obj M.Tm := by

theorem NaturalModelId.J_section [NaturalModelId Ctx] : J
(Ctx := Ctx) ≫ ε = 1 _ := by
```

The Groupoid Model of HoTT

The Hofmann-Streicher groupoid model (1995):

- Types A are groupoids.
- Terms $x : A$ are objects.
- Identity types $Id_A\, x\, y$ are hom-sets (discrete groupoids).
- Dependent types $(x : A \vdash B : Type)$ are fibrations of groupoids.
- The propositional truncation of a type A, is the groupoid with the same objects as A, but with a unique isomorphism between any pair of objects.
- The universe consists of discrete groupoids.
- The universe is univalent.

We can use the groupoid model for

- synthetic group theory by defining groups as pointed, connected groupoids,
- groupoid quotients,
- Eilenberg-MacLane spaces $K(G,1)$, and some basic cohomology,
- classifying spaces BG, and the theory of covering spaces,
- calculation of $\pi_1(S^1) = Z$ using univalence and circle induction,
- Joyal's combinatorial species,
- Rezk completion of a small category.

We are half way there on obtaining the groupoid model of HoTT in Lean4.

References

i. Awodey, S. (2017) Natural models of homotopy type theory, MSCS 28(2). arXiv:1406.3219

ii. Hofmann, M and Streicher, T (1996). The groupoid interpretation of type theory

iii. A Formalization of Polynomial Functors in Lean 4: `https://github.com/sinhp/Poly`

iv. Groupoid Model of HoTT in Lean 4: `https://github.com/sinhp/groupoid_model_in_lean4/`